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A Study on Effect of Single Additive Outlier on Estimation of Value at Risk and Expected Shortfall under Peaks over Threshold Framework

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ABSTRACT

Value at Risk (VaR) is one of the most popular measures of risk associated with financial instruments. The generalized Pareto distribution (GPD) is useful in modeling values exceeding a high threshold in estimation of VaR and Expected Shortfall (ES). It has been observed that none of the existing methods for estimating parameters of GPD performs uniformly better than others (*P. Z. Bermudez and S. Kotz, 2010*), and even if one develops a method, it would suffer from heavy computational requirements. Recently, *P.Chen et.al (2017)* proposed a method of estimating parameters of GPD based on minimum distance approach and M-estimation. In literature survey it has been observed that a study on effect of outliers on estimation of parameters of GPD has not been carried out, though presence of outliers is common in financial data. In this regard this paper focuses on the effect of outliers on estimators of parameters of GPD, on estimators of VaR and estimators of ES under Peaks over Threshold (PoT) framework obtained from different methods. A simulation study is carried out to compare performances of five robust and seven non-robust methods for estimating parameters of GPD, VaR and ES in the presence of a single additive outlier under PoT set up and it is found that robust methods suggested by *P.Chen et.al* (*2017*) for estimating GPD parameters perform on par with other estimators when shape parameter k > 0 (for thin tailed distributions), in estimating VaR in presence of outlier, especially when sample size is moderately large.

1. Introduction

1.1: Background

Value at risk (VaR) is used widely in financial industry by all stake holders, like investors, portfolio managers, rating agencies and regulators. It indicates the maximum amount that an investor may loose over a given time horizon and with a given probability. It is commonly used, since it is easy to understand and it is reported as a single number that represents potential losses with some confidence level. There are several methods available in literature (*Jorion, 2001)* for estimating VaR and one among them is based on extreme value theory (*McNeil, 1997*). The field of extreme values has attracted the attention of Statisticians, Engineers, and Economists in the last few decades and there are two widely used approaches to analyze extreme data *(Pickands, 1975; Galambos, 1981)*, namely, the block-maxima approach *(Beirlant et al. 1996)* and the peaks-over-threshold (PoT) approach *(Davison and Smith, 1990).* The first approach considers the distribution of the maximum order statistic, and the generalized extreme value distribution is then fitted to the series of extremal observations. But this approach does not consider all data points, as only one data point in each block is taken into account (*Fisher and Tippett, 1928).* The second approach extracts the peak values which exceed a certain threshold and in this method, the excess values over high threshold are modeled with generalized pareto distribution (GPD); *McNeil and Saladin (1997).*

1.2: Generalized Pareto Distribution and Peaks over Threshold (PoT) framework

Pickands (1975) and *Balkemaand-de-Haan (1974)* proved that the limiting distribution of exceedances (or peaks) of a random variable X over a sufficiently high threshold u is generalized Pareto distribution (GPD) with distribution function F(x) is given below

$$F(X|\mu,\sigma,k) = \begin{cases} 1-\left(1-k\frac{x-\mu}{\sigma}\right)^{1/k} & k \neq 0 \\ 1-exp\left(-\frac{x-\mu}{\sigma}\right) & k = 0 \end{cases}$$

Where, location $\mu\ (\mu \in R)$, scale $\sigma\ (\sigma > 0)$ and shape parameter $k\ (k \in R)$, for $k \leq 0$ the range of x is $\mu \leq x \leq \infty$, while for $k > 0$, $\mu \leq x \leq \mu + \frac{\sigma}{k}$. The distribution can be classified into three types depending on the shape parameter k; as heavy-tailed, medium-tailed and thin-tailed, according as k < 0, k = 0 and k > 0 respectively.

Under PoT framework, we can estimate extremes for arbitrary distributions, if threshold value is sufficiently high. But the choice of threshold is critical, as high threshold leads to high variance due to few exceedences, but not biased, and a low threshold would necessitate using samples that are no longer considered as being in the tails which leads to increased bias. Hence one has to balance between bias and precision in selecting threshold value *u*.

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In literature several threshold selection methods have been suggested; (*Embrechts et al. 1999b)*.

It is often seen that the number of exceedances is small in peaks-over-threshold approach and if there is an outlier, abnormally large value present in excedances, it may distort estimates. Hence, our objective of this study is to see the effect of outlier on various estimators and also to compare their performance in estimation of parameters of GPD, VaR and ES, through a simulation study using bias and root mean square error criterion in presence of outlier. So, for this purpose, in the next the section, we discuss various estimation procedures for parameters of GPD, estimation of VaR and ES under PoT framework. In Section 3, using a simulation study, we evaluate performance of estimation methods for parameters of GPD, VaR and ES using bias and root mean square error criteria. Finally in Section 4, the summary of results of the simulation study is presented.

2. Estimation Of Parameters Of Gpd, Value At Risk And Expected Shortfall

2.1: Literature review: Estimation of parameters of GPD

Several parameter estimation methods have been studied for generalized Pareto distribution in literature and various methods have been compared under different conditions for estimating the GPD parameters. However, there are no universally accepted methods for estimating GPD parameters. Even if few methods are better than others over certain range of shape parameter *k*, they suffer from various constraints and convergence problems (*P.Z Bermudeza & S. Kotz 2010, Part I & II)*. Among these, maximum likelihood method (MLE) is preferred, due to its asymptotic optimality properties and has been studied by *Davison (1984), Smith (1984, 1985), Grimshaw (1993). Hosking and Wallis (1987)* compared maximum likelihood estimates with method of moments (MOM) and probability weighted moment (PWM) estimates over small ranges of *k*, $|k| \le ½$ as it is common to observe k between -1 and ½ (*Zhang and Stephen, 2009*) and found that probability weighted method performs well for $0 \le k \le 1$ and very good for $k \le ½$. *Castillo and Hadi (1997)* introduced elemental percentile method (EPM) and compared it with the MOM and the PWM methods, using root mean square error criterion when $|k| \le 2$, through simulation study and showed that the PWM estimator performs well in small samples for $k \le ½$. *Zhang and Stephens (2009)* and *Zhang (2010)* developed empirical Bayes method (EBM) based on the likelihood and which uses a data-driven prior to estimate parameters of GPD. They showed that EBM performs better than MLE, MOM, PWM and Likelihood Moment Estimator (LME) with respect to bias and mean square error when $– ½ < k < ½$.

Luceño (2006) proposed estimators based on minimum distance approach by minimising the squared differences between empirical and model distribution functions, given in terms of various goodness- of-fit statistics, including the Cramer–von Mises statistic (CM), the Anderson– Darling statistic (AD) and the right-tail weighted Anderson– Darling statistic (ADR) and compared few maximum goodness-of-fit (MGF) estimators with Quasi Maximum Likelihood (QML), MLE, MOM, PWM and EPM over *k* = -2, -1, 0, 1, 2. Recently *P.Chen et. al (2017)* proposed two new robust estimators for the GPD parameters using the minimum distance approach and M-estimation, where the Tuckey biweight function is used as the distance measure which minimizes the distance between the empirical distribution and the family of GPDs. They compared proposed methods with Maximum Likelihood (ML), the Elemental Percentile Method (EPM), and a method proposed by *Zhang (2010)* (EBM). They claim that as the distance measure is borrowed from robust estimation, the proposed methods are robust to outlier contamination and the breakdown point is as high as 50%. It is seen that many simulation studies for comparing different methods of estimation of GPD parameters, have been conducted by many authors but these simulation studies are somewhat difficult to compare, as they have been performed under different conditions. Moreover, some of the methods have never been compared via a simulation study. For more details refer *Bermudeza & S. Kotz (2010) part I & II*.

As our objective is to study the effect of outliers on estimation of VaR, we consider comparison of some robust methods with some traditional methods available in literature through a simulation study. Hence, we considered five robust methods and seven non robust methods which are available in R-environment. Among these three robust method are based on minimum distance approach (PZ, WPZ, MGF)

Table 1: List of estimation methods considered in the study

Robust Methods	Non Robust Methods
1. Robust methods proposed *P.Chen et.al* (2017), (PZ and WPZ)	5. PICKANDS (Pickands) *Pickands, J. (1975)*
2. Median Estimator, (MED), *Peng and Welsh, (2001)*	6. Maximum Likelihood Estimator, (MLE), *Smith (1984).*
3. Minimum Density Power Divergence, (MDPD), *Juárez and Schucany (2004)*	7. Maximum Penalized Likelihood, (MPLE), *Coles and Dixon (1999)*
4. Maximum Goodness-of-fit Estimators, (MGF), (Based on Anderson Darling statistic) *Alberto Luceño* (2006)	8. Probability Weighted Moments,(PWMU, PWMB), *Hosking and Wallis (1987)*
	9. Likelihood Moment Estimator, (LME), *Zhang (2007)*
	10. Empirical Bayes Method, (*EBM or* ZJ), *Zhang (2010),*

2.2. Value at Risk and Expected Shortfall

Generally returns on investments on a portfolio follow normal distribution and as a result, under typical conditions, VaR is thought to be almost as effective as expected shortfall (ES) in capturing risk. However, the financial crisis in 2007 highlights the importance of measuring the risk associated with non-normal returns. In this connection we are using generalized Pareto distribution to model returns exceeding a sufficiently high threshold value, in case of violation of normality assumption and for heavy tailed distribution. The procedure is to estimate parameters of GPD and then use them for estimation of VaR and ES.

Value at Risk (VaR)

If X denotes the return random variable with distribution function $F_X(x)$, then for each $\alpha \epsilon (0,1)$, the VaR with 100 (1–α) % confidence coefficient is defined as:

$$VaR_{(\alpha)}(X) = \inf(x: P(X \le x) > \alpha)$$
$$= \sup(x: P(X < x) \le \alpha)$$

$If\ X \sim GPD\ (\mu, \sigma, k)$, then $VaR_{(\alpha)}(X)$ under PoT framework is given by (Ref: Mc. Neil. A., 1998)

$$VaR(\alpha) = u + \frac{\sigma}{k}\left[\left[\frac{n}{N_u}(1-\alpha)\right]^{-k} - 1\right]$$

As value at risk is not a linear function of parameters of GPD, there is a need for studying estimation of VaR, especially in the presence of outliers. The main drawback with the use of VaR as a risk measure is that, it does not respond to losses exceeding the confidence level, as a result it cannot capture the risk associated with the shape of the distribution beyond the confidence level. *Artzner et al. (1997)* propose the use of expected shortfall as an improvement on VaR.

$$ES_{(\alpha)}(X) = E\left(X \mid X > VaR_{(\alpha)}(X)\right)$$

$$ES(\alpha) = \frac{VaR(\alpha)}{1-k} + \frac{\sigma - ku}{1-k}$$

This is expected value of return random variable beyond value at risk. So we consider comparison of twelve methods in estimating parameters of GPD, VaR and ES. As our main interest lies in effect of outlier, we consider comparisons of methods both in the absence and in the presence of a single outlier under PoT framework.

Simulation Study

3.1: Introduction

In this section, the aim is to compare few robust methods with non-robust methods for estimating parameters of GPD, estimators of VaR and ES. Hence, a comprehensive simulation study is carried out to compare performance of the estimators as referred in Table 1, which includes recently proposed methods by *P.Chen et. al (2017)*. In this study location parameter is set at 0 and scale parameter is set at 1, as simulation results are invariant of scale parameter *(Hosking & Wallis, 1987)*. In practice the value of shape is commonly observed between -1 and ½ (*Zhang and Stephens, 2009*) and also it is not uncommon to observe shape parameter $k \geq$ ½ (infinite variance) *(Castillo et. al 2005)*, due to violation of normality assumption (heavy tailed distribution), therefore we restrict our attention to the case of shape (k) values between -1 and +1.

In this study, bias and root mean square errors are computed in estimating VaR and ES using all methods considered in this study, at different confidence levels ($1-\alpha$ = 0.95, 0.98 and 0.99) under PoT setup. Under this setup the sizes of exceedances are usually small due to critical choice of threshold, therefore in order to have some exceedances, i.e., n = 20, 40 and 80, samples of 1,000 random observations are generated from GPD at p = 0.02, 0.04 and 0.08 respectively.

For each combination of sample size (n), shape (k) and confidence level ($1-\alpha$), bias and root mean squared error of the estimators are obtained based on 10,000 Monte Carlo replications using R-software.

$$Bias = E(\hat{\theta} - \theta)\,; \quad RMSE = \sqrt{E\left[\left(\hat{\theta} - \theta\right)^2\right]}\,;$$

As a single abnormal large value may affect the precision of the estimators, performance of these methods are also compared after introducing a single additive outlier in estimating parameter of GPD, VaR and ES.

3.2 Algorithm

Step 1: Generated a random sample of size 1,000 observations from GPD at location μ = 0, scale σ = 1 and shape = k

Step 2: Select a threshold value u, taken to be $(1 - p)^{th}$ sample quantile

Step 3: Compute true Value at Risk (VaR) and true Expected Shortfall (ES) for fixed values of parameters of GPD, threshold u and confidence level *(1- α)*

$$VaR(\alpha) = u + \frac{\sigma}{k}\left[\left[\frac{n}{N_u}(1-\alpha)\right]^{-k} - 1\right]$$

$$ES(\alpha) = \frac{VaR(\alpha)}{1-k} + \frac{\sigma - ku}{1-k}$$

Step 4: Compute mean, standard deviation of observations obtained in step 1 and replace a random observation by an additive outlier (mean+(5*StdDev)) to obtain contaminated data

Step 5: Obtain observations above threshold u and generate exceedances $(X - u)$

Step 6: Fit GPD for exceedances (for both non-contaminated and contaminated data) using all methods and estimate σ (scale), k (shape), VaR and ES.

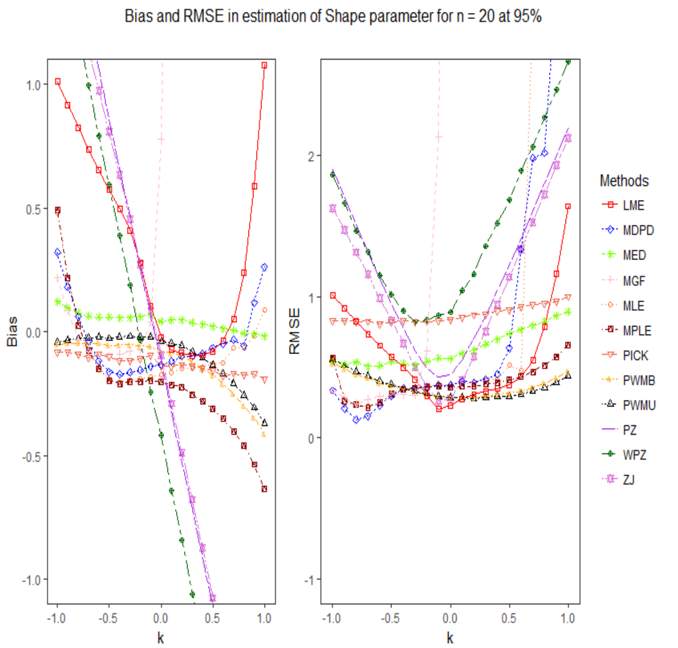
Step 7: Above steps from 1 to 6 are repeated 10,000 times and compute average bias and square root of average of squared error (RMSE) for scale, shape, VaR and ES for both contaminated and non-contaminated data.

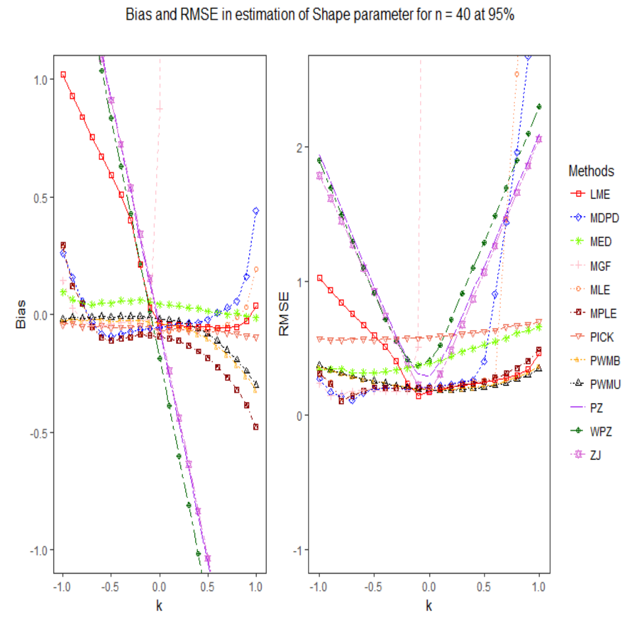
3.3: Results and findings

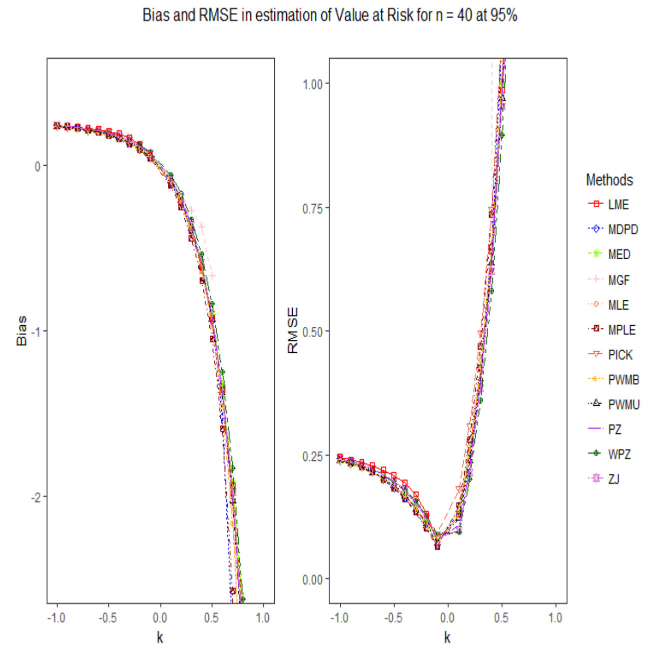
It is observed that the performance of different methods in estimation of parameters of GPD depends on the sample size and the shape of the sampling distribution. Also we found that there is no one estimator, which stands out as being the best in all situations. Following are some of the general findings of the simulation study.

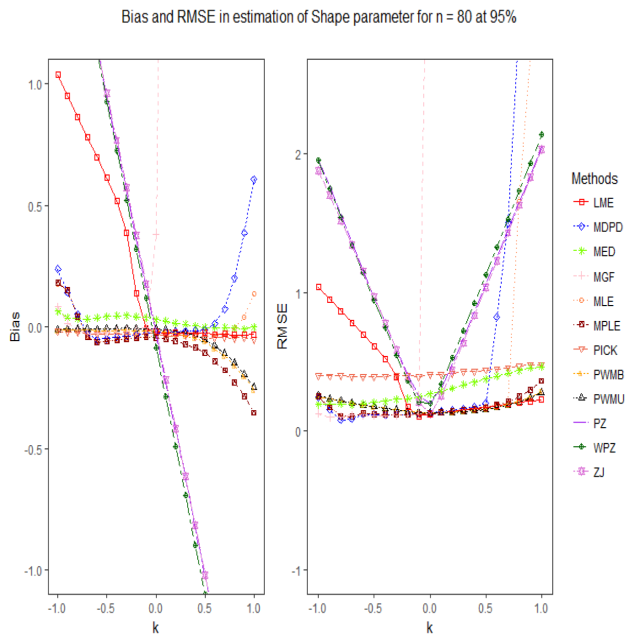
- RMSE in estimation of VaR increases considerably with increase in confidence level $(1-\alpha)$ for all methods.
- Distribution of RMSE in estimation of VaR is found to be asymmetric over range of $-1 < k < 1$ for a given shape and confidence level for all methods.
- Single additive outlier affected estimation of Value at Risk and Expected shortfall even when sample size is moderately large.

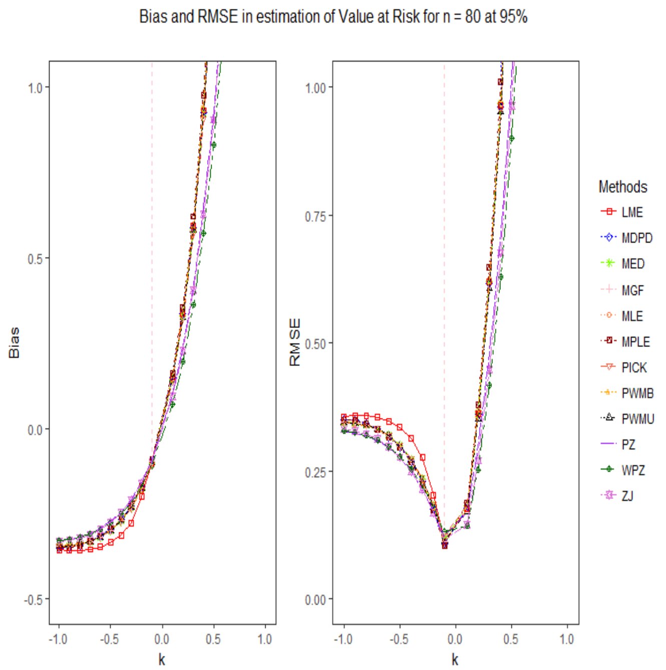
Bias and root mean square error values in estimation of shape, VaR and ES are computed at 95%, 98% and 99% confidence levels. For space constraint, only RMSE values obtained at sample size n = 80 is reported in Table 2.1 to Table 4.2. However, graphs representing bias and RMSE values for estimating shape and VaR are reported for n = 20, 40 and 80.

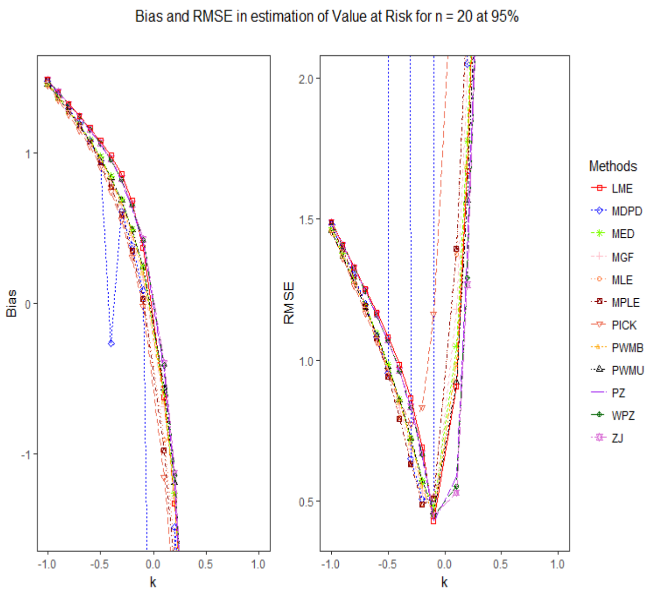












Detailed summary of simulation study in absence of outlier is given below

Shape Parameter

- PWM methods are consistent in estimating shape parameter, but however bias increases sharply when k > ½ and PWM underestimates shape over the range of -1<k< 1.
- As sample size increases MPLE, LME & MDPD were performing on par with PWM method but RMSE of MDPD increases sharply, when k > ½, in estimating shape parameter.

Scale Parameter

- As sample size increases all methods were performing equally well in estimation of scale
- Rate of change in RMSE for scale is more when k > 0, as compared to k < 0 for all estimators

Table 2.1. RMSE in estimating shape (without outlier) when n = 80 (p= 0.08, N = 1,000) at 95% confidence level.

k	**-1**	**-0.9**	**-0.8**	**-0.7**	**-0.6**	**-0.5**	**-0.4**	**-0.3**	**-0.2**	**-0.1**	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1**
ZJ	1.87	1.70	1.52	1.34	1.15	0.96	0.77	0.58	0.39	0.21	0.12	0.26	0.44	0.63	0.83	1.03	1.23	1.43	1.63	1.83	2.03
PZ	1.95	1.76	1.56	1.36	1.16	0.97	0.77	0.58	0.41	0.25	0.20	0.31	0.48	0.67	0.86	1.05	1.25	1.45	1.65	1.85	2.05
WPZ	1.95	1.75	1.54	1.34	1.14	0.94	0.74	0.55	0.36	0.21	0.20	0.34	0.53	0.72	0.92	1.12	1.32	1.52	1.73	1.93	2.14
MLE	0.25	0.17	0.10	0.10	0.12	0.12	0.12	0.12	0.12	0.13	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.20	1.65	2.72	4.17
PWMU	0.25	0.23	0.22	0.20	0.19	0.18	0.16	0.15	0.14	0.14	0.13	0.13	0.13	0.14	0.15	0.16	0.17	0.18	0.21	0.24	0.28
PWMB	0.25	0.23	0.22	0.20	0.19	0.18	0.16	0.15	0.14	0.14	0.13	0.13	0.14	0.14	0.15	0.16	0.17	0.19	0.22	0.25	0.29
PICK	0.40	0.40	0.40	0.39	0.39	0.39	0.40	0.40	0.40	0.40	0.41	0.41	0.42	0.43	0.43	0.44	0.45	0.46	0.47	0.47	0.48
MED	0.19	0.20	0.20	0.19	0.20	0.21	0.21	0.23	0.24	0.25	0.27	0.29	0.31	0.33	0.35	0.38	0.40	0.42	0.44	0.45	0.47
MDPD	0.25	0.15	0.08	0.09	0.12	0.12	0.12	0.12	0.12	0.13	0.13	0.14	0.15	0.16	0.17	0.20	0.82	1.50	2.92	3.73	4.69
LME	1.04	0.95	0.86	0.78	0.70	0.61	0.52	0.39	0.17	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.21	0.22	0.23
MPLE	0.25	0.17	0.11	0.11	0.13	0.12	0.12	0.12	0.12	0.12	0.13	0.14	0.14	0.15	0.16	0.17	0.19	0.21	0.25	0.30	0.36
MGF	0.13	0.10	0.11	0.12	0.12	0.11	0.11	0.12	0.12	0.13	5.25	17.39	29.73	39.67	48.33	51.87	52.62	50.51	47.40	44.33	38.75

Table 2.2. RMSE in estimating shape (with outlier) when n = 80 (p= 0.08, N = 1,000) at 95% confidence level.

k	**-1**	**-0.9**	**-0.8**	**-0.7**	**-0.6**	**-0.5**	**-0.4**	**-0.3**	**-0.2**	**-0.1**	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1**
ZJ	0.82	0.75	0.67	0.60	0.54	0.47	0.41	0.35	0.28	0.17	0.12	0.25	0.44	0.63	0.83	1.04	1.24	1.45	1.66	1.86	2.08
PZ	1.05	0.96	0.88	0.80	0.72	0.65	0.57	0.47	0.36	0.24	0.21	0.33	0.50	0.69	0.89	1.09	1.29	1.49	1.69	1.90	2.11
WPZ	1.05	0.96	0.88	0.80	0.70	0.54	0.46	0.39	0.30	0.19	0.21	0.36	0.54	0.74	0.94	1.14	1.35	1.56	1.77	1.98	2.19
MLE	1.16	1.03	0.90	0.77	0.64	0.50	0.35	0.21	0.10	0.10	0.13	0.15	0.16	0.17	0.18	0.19	0.20	0.67	1.56	2.79	4.41
PWMU	0.93	0.75	0.60	0.46	0.35	0.27	0.21	0.17	0.14	0.13	0.13	0.13	0.13	0.14	0.15	0.16	0.17	0.19	0.21	0.24	0.28
PWMB	0.92	0.75	0.59	0.46	0.35	0.26	0.20	0.16	0.14	0.13	0.12	0.13	0.13	0.14	0.15	0.16	0.18	0.20	0.22	0.25	0.29
PICK	0.40	0.40	0.40	0.39	0.39	0.40	0.40	0.40	0.40	0.39	0.41	0.41	0.42	0.43	0.43	0.44	0.45	0.46	0.47	0.48	0.49
MED	0.91	0.80	0.69	0.57	0.46	0.36	0.27	0.23	0.23	0.26	0.28	0.30	0.32	0.34	0.36	0.38	0.40	0.43	0.44	0.46	0.47
MDPD	1.03	0.92	0.81	0.70	0.58	0.45	0.32	0.20	0.11	0.11	0.13	0.14	0.16	0.17	0.18	0.27	0.86	1.58	2.85	3.35	4.63
LME	1.07	0.94	0.82	0.70	0.60	0.52	0.41	0.27	0.12	0.11	0.12	0.13	0.15	0.16	0.17	0.18	0.20	0.21	0.23	0.24	0.25
MPLE	1.15	1.03	0.90	0.77	0.63	0.49	0.35	0.21	0.10	0.09	0.12	0.14	0.15	0.16	0.17	0.17	0.19	0.21	0.24	0.29	0.35
MGF	5.67	9.62	15.77	22.65	15.95	2.15	0.49	0.30	0.14	0.10	4.70	15.47	28.30	37.81	46.86	50.67	52.13	49.84	47.54	43.42	39.46

Table 3.1. RMSE in estimating Value at Risk (without outlier) when n = 80 (p= 0.08, N = 1,000) at 95% confidence level.

k	**-1**	**-0.9**	**-0.8**	**-0.7**	**-0.6**	**-0.5**	**-0.4**	**-0.3**	**-0.2**	**-0.1**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1**
ZJ	0.33	0.33	0.32	0.31	0.30	0.28	0.25	0.21	0.17	0.12	0.15	0.27	0.45	0.68	0.96	1.36	1.82	2.45	3.21	4.18
PZ	**0.33**	**0.32**	**0.32**	**0.31**	**0.29**	**0.27**	**0.25**	**0.21**	0.17	0.12	0.17	0.30	0.48	0.71	1.01	1.40	1.88	2.52	3.30	4.28
WPZ	0.33	0.33	0.32	0.31	0.30	0.28	0.25	0.22	0.18	0.13	**0.14**	**0.25**	**0.42**	**0.63**	**0.90**	**1.27**	**1.70**	**2.29**	**3.00**	**3.90**
MLE	0.35	0.35	0.34	0.33	0.32	0.30	0.27	0.23	0.17	**0.11**	0.18	0.37	0.63	0.97	1.42	2.07	9.22	21.68	49.87	153.60
PWMU	0.34	0.34	0.34	0.33	0.32	0.30	0.27	0.23	0.18	0.11	0.17	0.35	0.61	0.95	1.41	2.07	2.94	4.19	5.92	8.42
PWMB	0.34	0.34	0.34	0.33	0.32	0.30	0.27	0.23	0.18	0.11	0.18	0.36	0.62	0.97	1.44	2.12	3.04	4.95	6.40	9.63
PICK	0.35	0.34	0.34	0.33	0.32	0.30	0.27	0.24	0.18	0.12	0.18	0.36	0.62	0.96	1.41	2.03	2.83	3.95	5.36	7.31
MED	0.34	0.34	0.34	0.33	0.32	0.30	0.27	0.24	0.18	0.12	0.18	0.36	0.62	0.97	1.43	2.05	2.86	3.99	5.43	7.38
MDPD	0.35	0.35	0.34	0.33	0.32	0.30	0.27	0.23	0.18	0.11	0.18	0.36	0.62	0.96	3.51	25.95	40.13	76.80	285.50	473.11
LME	0.36	0.36	0.36	0.35	0.35	0.33	0.31	0.28	0.20	0.11	0.18	0.36	0.62	0.97	1.42	2.05	2.87	3.98	5.43	7.36
MPLE	0.35	0.35	0.34	0.33	0.32	0.30	0.27	0.23	0.17	0.11	0.19	0.38	0.65	1.01	1.49	2.31	7.51	43.88	134.91	487.78
MGF	0.35	0.34	0.34	0.33	0.32	0.30	0.27	0.23	0.18	0.11	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN

Table 3.2. RMSE in estimating Value at Risk (with outlier) when n = 80 (p= 0.08, N = 1,000) at 95% confidence level.

k	-1	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
ZJ	0.36	0.36	0.36	0.35	0.34	0.33	0.30	0.25	0.19	0.12	0.16	0.29	0.48	0.71	1.01	1.40	1.87	2.50	3.26	4.21
PZ	0.35	0.35	0.35	0.34	0.33	0.30	0.27	**0.22**	**0.17**	0.12	0.17	0.30	0.48	0.72	1.01	1.40	1.88	2.51	3.27	4.25
WPZ	0.35	0.35	0.35	0.34	0.33	0.31	0.28	0.24	0.18	0.13	**0.15**	**0.26**	**0.43**	**0.65**	**0.92**	**1.29**	**1.72**	**2.30**	**3.00**	**3.88**
MLE	0.35	0.36	0.35	0.35	0.34	0.32	0.29	0.25	0.18	0.10	0.20	0.39	0.66	1.01	1.48	2.13	3.22	8.20	12.08	34.77
PWMU	0.35	0.35	0.34	0.34	0.32	0.30	0.27	0.23	0.18	0.11	0.18	0.37	0.64	0.99	1.47	2.16	3.09	9.02	6.35	9.45
PWMB	0.35	0.35	0.34	0.34	0.32	0.30	0.27	0.23	0.18	0.11	0.19	0.38	0.65	1.01	1.51	2.22	3.20	27.97	7.02	12.57
PICK	**0.34**	**0.34**	**0.34**	**0.33**	**0.32**	**0.30**	**0.27**	0.23	0.18	0.11	0.20	0.39	0.65	1.01	1.48	2.13	2.97	4.12	5.62	7.66
MED	0.35	0.35	0.34	0.34	0.32	0.30	0.28	0.24	0.18	0.11	0.19	0.38	0.64	0.99	1.46	2.10	2.92	4.07	5.54	7.53
MDPD	0.35	0.35	0.35	0.35	0.34	0.32	0.29	0.24	0.18	0.10	0.19	0.38	0.65	1.00	8.64	20.18	116.54	410.92	942.07	5572.07
LME	0.35	0.35	0.35	0.35	0.34	0.32	0.30	0.25	0.18	0.11	0.19	0.38	0.65	1.01	1.47	2.13	2.96	4.11	5.60	7.58
MPLE	0.35	0.36	0.35	0.35	0.34	0.32	0.29	0.25	0.18	**0.10**	0.21	0.40	0.68	1.06	1.55	3.47	11.12	2066.18	159.24	564.19
MGF	296	97.63	25.46	11.60	4.72	1.45	0.31	0.26	0.19	0.11	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN

Table 4.1. RMSE in estimating Expected Shortfall (without outlier) when n = 80 (p= 0.08, N = 1,000) at 95% confidence level.

k	-1	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
ZJ	152	12.9	2.03	**0.41**	**0.32**	**0.30**	**0.31**	**0.33**	**0.33**	0.36	**0.49**	**0.58**	**0.71**	**0.86**	**1.04**	**1.26**	**1.57**	**2.78**	9.11
PZ	207	108	50.7	9.32	1.01	0.50	0.42	0.44	0.45	0.49	0.68	0.81	0.98	1.17	1.41	1.66	1.98	3.03	**9.08**
WPZ	114	90.2	61.7	32.3	0.63	0.44	0.39	0.41	0.42	0.44	0.54	0.62	0.73	**0.86**	**1.04**	1.30	1.72	3.16	9.70
MLE	0.64	0.66	0.68	0.68	0.69	0.68	0.65	0.59	0.48	0.31	0.55	1.32	2.67	5.44	24.87	196.10	1185.7	3923.43	5564.92
PWMU	**0.63**	**0.65**	0.67	0.68	0.69	0.68	0.65	0.59	0.48	0.31	0.55	1.31	2.57	4.73	8.80	19.89	45.27	5281.53	421.82
PWMB	**0.63**	**0.65**	0.67	0.68	0.69	0.68	0.65	0.59	0.48	0.31	0.55	1.30	2.57	4.72	8.79	19.87	45.23	5277.10	421.47
PICK	0.63	0.65	0.67	0.68	0.69	0.67	0.67	0.63	0.94	1.04	15.7	74.2	383.8	351.90	1022.46	16102	1928.9	1810.31	22179.14
MED	0.63	0.65	0.67	0.68	0.68	0.67	0.64	0.59	4.80	2.21	8.09	30.3	300.6	157.29	8084.34	2496	2416.6	28000.47	4581.05
MDPD	0.64	0.66	0.68	0.68	0.69	0.68	0.65	0.59	0.48	0.31	0.56	1.36	2.82	36.48	171.90	496.30	5982.5	48437.22	30657780
LME	0.63	0.65	**0.66**	0.67	0.67	0.65	0.62	0.56	0.47	0.31	0.55	1.33	2.67	5.63	16.54	297.96	744.34	5681.67	6129.42
MPLE	0.64	0.66	0.68	0.68	0.69	0.68	0.65	0.59	0.48	0.31	0.53	1.26	2.44	4.29	7.15	12.45	50.43	12860.79	958.83
MGF	0.63	0.65	0.67	0.68	0.69	0.68	0.65	0.59	0.48	**0.30**	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN

Table 4.2. RMSE in estimating Expected Shortfall (with outlier) when n = 80 (p= 0.08, N = 1,000) at 95% confidence level.

k	-1	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
ZJ	0.64	0.66	0.68	0.69	0.68	0.66	0.61	0.50	**0.36**	0.32	**0.52**	**0.63**	0.77	0.93	1.12	1.35	**1.64**	**2.82**	**9.14**
PZ	0.64	**0.62**	**0.62**	**0.61**	**0.59**	**0.53**	**0.44**	**0.36**	0.38	0.47	0.67	0.79	0.96	1.13	1.36	1.59	1.93	3.06	9.22
WPZ	0.64	0.62	0.62	0.62	0.60	0.60	0.54	0.45	0.39	0.42	0.55	0.63	**0.75**	**0.89**	**1.07**	**1.33**	1.75	3.22	9.80
MLE	0.61	0.63	0.65	0.65	0.65	0.64	0.61	0.55	0.45	0.28	0.60	1.40	2.84	6.44	43.47	665.97	5.07E+10	8.76E+18	1.41E+16
PWMU	**0.61**	0.63	0.65	0.66	0.66	0.65	0.62	0.55	0.45	0.28	0.60	1.38	2.68	4.92	9.28	21.72	50.52	6044.23	481.24
PWMB	**0.61**	0.63	0.65	0.66	0.66	0.65	0.62	0.56	0.45	0.28	0.60	1.37	2.67	4.92	9.27	21.70	50.48	6039.22	480.84
PICK	0.63	0.65	0.67	0.68	0.68	0.67	0.64	0.66	1.70	2.18	16.24	73.44	99.74	263	773	2164	1683	1690	1892
MED	0.61	0.63	0.64	0.64	0.64	0.63	0.61	0.54	1.20	2.11	8.73	28.51	91.23	389	5319	810	2394	1698	20742
MDPD	0.62	0.64	0.65	0.66	0.66	0.64	0.61	0.55	0.44	**0.27**	0.62	1.45	3.17	10.10	183.13	3.60E+08	2.95E+11	2.54E+32	1.19E+14
LME	0.61	0.63	0.64	0.65	0.65	0.64	0.61	0.55	0.44	0.28	0.61	1.41	2.83	6.11	73.45	139.85	2410.42	22290.93	5552.61
MPLE	0.61	0.63	0.65	0.65	0.66	0.64	0.61	0.55	0.45	0.28	0.58	1.34	2.56	4.50	7.56	26.56	89.03	15734.80	1194.04

Value at Risk

- RMSE in estimating VaR is increasing as confidence level $(1 - \alpha)$ increases for all sample sizes.
- When sample size is small (n = 20, 40) all methods overestimates VaR, when $k < 0$, and underestimates VaR when $k > 0$.
- But as sample size increases (n=80), all methods underestimates VaR when $k < 0$, and overestimates VaR when $k > 0$.
- RMSE in estimating VaR for all methods increases rapidly when $k > 0.2$.
- Robust methods PZ and WPZ methods based on minimum distance and M-estimation are performing well in estimation of VaR, when $k \le 0.2$ and $k > 0.1$ respectively for large sample size (n=80).

Expected Shortfall

- As sample size increases PZ, WPZ and EBM methods are performing equally well in estimating expected shortfall
- But however, EBM is better than others in estimating ES based on RMSE when $-0.7 < k < 0$ for large sample size. (n=80)

We finally conclude that no estimators are performing better than others in estimating shape, VaR and ES over the range of $-1 < k < 1$ in absence of outlier.

3.4 Summary of simulation study

Table 5. Estimators of shape parameter based on minimum root mean square error in this simulation study

Sample size	Non Contaminated	Contaminated
n=20	MDPD ($-1 < k < -0.5$), LME ($-0.2 < k < 0.1$), PWMU ($0.1 < k < 1$)	ZJ ($-1 < k < -0.2$), LME ($-0.2 < k < 0.1$), PWMU ($0.1 < k < 1$)
n = 40	MGF($-0.5 < k < -0.1$), LME($k = -0.1, 0$), PWMU ($0.1 < k < 1$)	ZJ($-0.9 < k < -0.4$), LME($k = -0.1$), PWMU ($0 < k < 1$)
n = 80	MGF($-0.6 < k < -0.2$), LME($k = -0.1$), ZJ($k=0$), PWMU ($0.1 < k < 0.7$), LME($0.8 < k < 1$)	Pick ($-1 < k < -0.7$), PWMB($-0.6 < k < -0.3$), MPLE ($k = -0.2, -0.1$), LME($k=0$), PWMU($0 < k < 1$)

Table 6. Estimators of VaR at 95% based on minimum root mean square error in this simulation study

Sample size	Non Contaminated	Contaminated
n=20	PICKANDS ($k \le -0.6$) MPLE ($-0.5 \le k \le -0.2$) PZ ($k = -0.1$) ZJ($0 < k \le 0.3$) PWMU ($k=0.4, 0.5$) LME ($k \ge 0.6$)	MGF ($k \le -0.4$) MED ($k = -0.3$) MPLE ($k = -0.2$) LME ($k = -0.1$) ZJ($0 < k \le 0.3$) PWMU ($k=0.4$) LME ($k \ge 0.5$)
n = 40	PICKANDS ($k \le -0.8$) MGF ($k = -0.7$) MPLE ($k = -0.6$) MLE ($-0.5 \le k < -0.2$) MPLE ($k = -0.1$) WPZ ($k > 0$)	PICKANDS ($k < -0.2$) MPLE ($k = -0.1$) WPZ ($k > 0$)
n=80	PZ ($k \le 0.2$) MLE ($k = -0.1$) WPZ($k > 0$)	PICKANDS ($k < -0.4$) PZ ($k = -0.3, -0.1$) MPLE ($k = -0.1$) WPZ ($k > 0$)

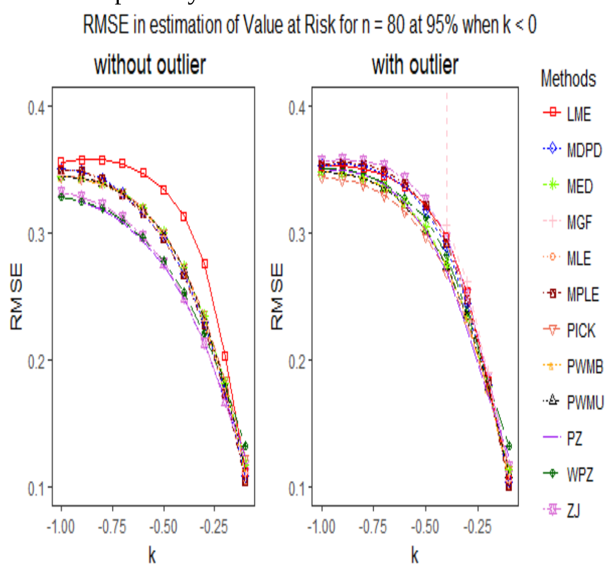
Table 7. Estimators of ES at 95% based on minimum root mean square error in this simulation study

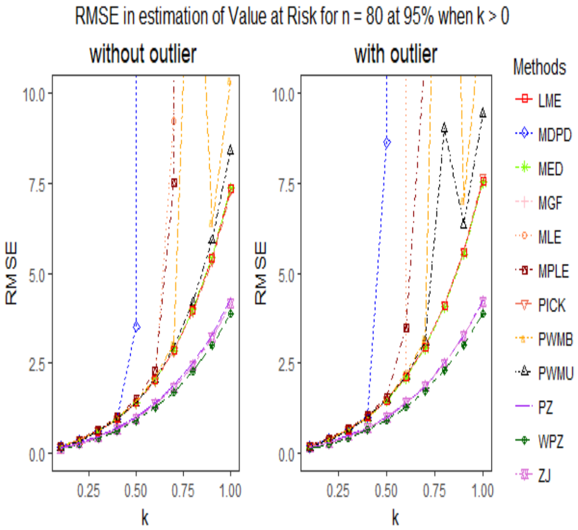
Sample size	Non Contaminated	Contaminated
n=20	PICKANDS ($k = -1, -0.9$) PWM ($-0.9 \le k \le -0.6$) MLE ($k = -0.5, -0.4, -0.3$) LME ($k = -0.2, -0.1$) PWM ($k = 0.1$) ZJ($0.2 \le k \le 0.9$)	PICKANDS ($-1 \le k \le -0.8$) ZJ ($-0.7 \le k \le -0.3$) LME ($k = -0.2, -0.1$) PWMU ($k = 0.1$) ZJ($0.2 \le k \le 0.9$)
n = 40	PWM ($-1 \le k \le -0.7$) ZJ ($k=-0.6$) LME ($k=-0.5$) ZJ($k=-0.4, -0.3, -0.2$) LME ($k=-0.1$) MPLE ($k=0.1$) WPZ ($0.2 \le k \le 0.9$) ZJ ($k=1$)	PWM ($-1 \le k \le -0.4$) PZ ($k=-0.3$) MGF ($k=-0.2, -0.1$) MPLE ($k=0.1$) WPZ ($0.2 \le k \le 0.9$) ZJ ($k=1$)
n=80	PWM($k = -1, -0.9$) LME ($k=-0.8$) ZJ ($-0.7 \le k \le -0.2$) MGF($k=-0.1$) ZJ ($0.1 \le k \le 0.8$) PZ ($k = 0.9$)	PWM($k = -1$) PZ ($-0.9 \le k \le -0.3$) ZJ ($k=-0.2$) MDPD($k=-0.1$) ZJ ($k=0.1, 0.2$) WPZ ($0.3 \le k \le 0.6$) ZJ ($0.7 \le k \le 0.9$)

Following are the observations from above table

- It is clear that estimators that perform well for VaR are different from those of shape parameter, which is evident due to nonlinear relationship between VaR and shape parameter
- It is also observed that WPZ method was performing better than all methods, in estimation of value at risk when $k > 0$, for sufficiently large sample size, in both the cases.
- WPZ, PZ and EBM methods are performing equally well in estimating ES under following cases when i) $k > 0$ and n = 40 and ii) $-1 < k < 1$ and n = 80.
- EBM (ZJ) among them is performing better than PZ and WPZ when $-0.7 < k < -0.2$ in absence of outlier when n = 80. And WPZ, PZ and EBM are performing better than others in estimating ES when $k > 0$ for both datasets at n = 80.

Following graphs are reported below to facilitate comparison of effects of introducing single additive outlier in estimation of Value at Risk for negative and positive values of shape parameter separately at n=80.





From above plots we observed that PZ and WPZ are performing better than others in estimating VaR in absence of outlier, but for contaminated data robust methods PZ and WPZ were not performing well, even when sample size is large (n=80) for k < 0. Surprisingly for k > 0, these robust methods are performing better than others in presence of outlier when sample is large (n=80) when k > 0.

3. Concluding Remarks

The main objective of this study is to investigate the performances of some estimators of parameters of GPD, Value at Risk and Expected Shortfall under PoT framework in the presence of an outlier when sample size is small and moderately large. With this aim, some widely used GPD estimators referred in Table-1 are compared using bias and root mean square error criteria and VaR and ES so obtained from these estimators are also compared under same criterion. We observe that all methods considered in simulation study are affected by the presence of single additive outlier and further it is found that PWM based methods are consistent in estimating shape parameter when sample size is small (n=20) which supports finding reported in *Hosking and Wallis (1987)* and *Castillo and Hadi (1997)*, and in absence of outlier, we observe that no estimator is performing better than others in estimating shape parameter over the range of $-1 < k < 1$. We also notice that estimators of VaR and ES are affected more by the presence outliers as compared to effect on GPD estimators.

We note that robust method WPZ having least root mean square error in estimation of VaR among all estimators for k > 0 in presence and absence of outliers when sample size is moderately large. Also in estimation of ES, EBM (ZJ) estimator is having minimum root mean square error for most of the values of k in the range -1 to 1 as compared to that of other estimators in absence and in presence of outlier when sample size is moderately large. Hence, based on this study, we conclude that one can use robust method WPZ for estimating Value at Risk (extreme quantiles) and ZJ method for estimating ES without looking for outliers when the size of exceedances is large under peaks over threshold framework. We believe that above findings and conclusion facilitate investors/practitioners in selecting appropriate method in calculating risk measures wherever outliers are likely to occur.

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