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Common Fixed Point Theorems for Four Mappings in Complete 2-Banach Space

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ABSTRACT
This paper deals with few fixed point theorem for four mappings and some results on 2-Banach space.

Introduction

In 1976 Iseki [1] introduced some fixed point theorems in Banach space. In 1993, Khan introduced involutions with fixed points in 2-Banach space. In the present paper I establish some common fixed point results for four mappings in 2-Banach space which mainly generalize the results of Amalendu Choudhury and T. Som and V.H. Badsha , Rekha Jain and saurabh Jain.

Preliminary definitions and results:

Let X be a linear space and $\|.,.\|$ be a real valued function defined on X satisfying the following conditions:

(i) $\|x,y\| = 0$ if and only if x and y are linearly dependent.

(ii) $\|x,y\| = \|y,x\|$ for all x,yϵX

(iii) $\|x,y+z\| \le \|x,y\| + \|x,z\| forall x,y,z \in X$

$\|.,.\|$ is called a 2-norm and the pair $(X,\|.,.\|)$ is called a linear 2-normed space.

Basic properties of the 2-norms are that they are non-negative and $\|x,y+ax\| = \|x,y\| \forall x,y \in X$ and all real number a.

A sequence $\{x_n\}$ in a linear 2-normed space (X,$\|.,.\|$) is called a cauchy sequence if $\lim_{m,n\to\infty}\|x_m - x_n,y\| = 0 \ \forall$ y in X.

A sequence $\{x_n\}$ is a linear 2-normed space (X,$\|\cdot,\cdot\|$) is said to be convergent to a point x in X if $\lim_{m\to\infty}\|x_m - x,y\| = 0 \ \forall y in X.$

A linear 2-normed space (X,$\|\cdot,\cdot\|$) in which every Cauchy sequence is convergent is called a 2-Banach space.

Let X be a 2-Banach space and T be a self mapping of X. T is said to be continuous at x if for any sequence $\{x_n\}$ in X with $x_n \to x$ then $Tx_n \to Tx$.

Let X be a 2-Banach space. T and S are said to be weakly compatible if they commute at their coincidence points. i.e., Tx = Sx for some x$\in X \Rightarrow TSx = STx.$

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Main Results:

Let X be complete 2-normed linear space such that 2-norm satisfies the $\alpha -$property with $\alpha > 0$. Let A,B,S and T be self mappings of X satisfying the following conditions:

(i) $A(x) \subseteq T(X), B(X) \subseteq S(X) and T(X) or S(X)$ is a closed subset of X.

(ii) The pairs (A,S) and (B,T) are weakly compatible.

$For all$ x,yϵX, $\|Ax, By\| \leq k_1[(\|Sx, Ty\| \cdot \|Ax, Sx\| \cdot \|By, Ty\|)] + k_2\left[\left(\|Sx, Ty\| \cdot \frac{\|Sx,By\|+\|Ax,Ty\|}{2}\right) \cdot \|By, Ty\|\right]$ where k_1 , $k_2 > 0 \; and \; 0 < (k_1 + k_2) < 1$ the A,B,S and T have a unique common fixed point in X.

Proof:

Let x_0 be an arbitrary point in X. By (i) we can define inductively a sequence $\{y_n\}$ in X such that $y_{2n} = Ax_{2n} = Tx_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$ for n=0,1,2,…

We claim that the sequence $\{y_n\} is a$ Cauchy sequence

$$\|y_{2n}, y_{2n+1}\| = \|Ax_{2n}, Bx_{2n+1}\|$$

$$\leq k_1[(\|Sx_{2n}, Tx_{2n}\| \cdot \|Ax_{2n}, Sx_{2n}\|) \cdot \|Bx_{2n+1}, Tx_{2n+1}\|]+$$

$$k_2\left[\left(\|Sx_{2n}, Tx_{2n+1}\| \cdot \frac{\|Sx_{2n}, Bx_{2n+1}\| + \|Ax_{2n}, Tx_{2n+1}\|}{2} \cdot \|Bx_{2n+1}, Tx_{2n+1}\|\right)\right]$$

$$\leq k_1[(\|y_{2n-1}, y_{1n}\| \cdot \|y_{2n}, y_{2n-1}\|) \cdot \|y_{2n+1}, y_{2n}\|]$$

$$+k_2\left[\left(\|y_{2n-1}, y_{2n}\| \cdot \frac{\|y_{2n-1},y_{2n+1}\|+\|y_{2n},y_{2n-1}\|}{2} \cdot \|y_{2n+1}, y_{2n}\|\right)\right]$$

Let $d_n = \|y_n, y_{n+1}\|$

$$\therefore d_{2n} \leq k_1[(d_{2n-1} \cdot d_{2n-1}) \cdot d_{2n}] + k_2\left[\left(d_{2n-1} \cdot \frac{d_{2n-1} + d_{2n-1}}{2}\right) \cdot d_{2n}\right]$$

$$\leq \alpha k_1 max\{d_{2n-1}, d_{2n-1}, d_{2n}\} + \alpha k_2 max\{d_{2n-1}, d_{2n-1}, d_{2n}\}$$

Suppose $d_{2n} > d_{2n-1}$

$d_{2n} \leq \alpha k_1 d_{2n} + k_2 \alpha d_{2n} = (k_1 + k_2)\alpha d_{2n}$ which is a contradiction.

Hence, $d_{2n-1} > d_{2n} \Rightarrow d_{2n} \leq d_{2n-1}$

Similarly, $d_{2n+1} \leq d_{2n}$

$$\therefore d_n \leq d_{n-1} for n = 1, 2, \dots$$

Using above, $d_n \leq \alpha(k_1 + k_2)d_{n-1} = kd_{n-1}, where \alpha(k_1 + k_2) = k < 1$

$$\therefore d_n \leq kd_{n-1} \leq k \cdot kd_{n-2} \leq \cdots \leq k \cdot k \cdot \ldots \cdot kd_0 \; (n times)$$

$$\therefore d_n \leq kd_{n-1} \leq k^2 d_{n-2} \leq \cdots \leq k^n d_0$$

That is,

$$\|y_n, y_{n+1}\| \leq k^n \|y_0, y_1\| \to 0 \; as n \to \infty$$

If $m > n, \|y_n, y_m\| \leq \|y_n, y_{n+1}\| + \|y_{n+1}, y_{n+2}\| + \ldots + \|y_{m-1}, y_m\|$

$$\leq k^n\|y_0,y_1\| + k^{n+1}\|y_0,y_1\| + \ldots + k^{m-1}\|y_0,y_1\| = (k^n + k^{n+1} + \ldots + k^{m-1})\|y_0,y_1\|$$

$$= \frac{k^n}{1-k}\|y_0,y_1\| \to 0 \; as m, n \to \infty$$

It follows that $\{y_n\}$ is a Cauchy sequence and by the completeness of X, $\{y_n\}$ converges to $y \in X$.

$$\therefore \lim_{n\to\infty} y_n = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = y.$$

Assume that T(X) is a closed subset of X.

Then there exists $v \in X$ such that $Tv = y$.

If $Bv \neq y$, then by using (iii), we obtain $\|As_{2n}, Bv\| \leq k_1[(\|Sx_{2n}, Tv\| \cdot \|Ax_{2n}, Sx_{2n}\|) \cdot \|B_v, Tv\|]$ + $k_2\left[\left(\|Sx_{2n}, Tv\| \cdot \frac{\|Sx_{2n},Bv\|+\|Ax_{2n},Tv\|}{2}\right) \cdot \|Bv, Tv\|\right]$

As n$\to \infty$, we get

$$\|y, Bv\| \leq k_1[\|y, Tv\| \cdot \|y, y\| \cdot \|Bv, Tv\|] + k_2\left[\left(\|y, Tv\| \cdot \frac{\|y, Bv\| + \|y, Tv\|}{2}\right) \cdot \|Bv, Tv\|\right]$$

$$< (k_1 + k_2)\alpha\|Bv, y\|$$

It follows that Bv = y = Tv.

Since B and T are weakly compatible, we have BTv = TBv and so By = Ty.

If y $\neq$ By by (iii) we get

$$\left\|Ax_{2n}, B_y\right\| \leq k_1[(\|Sx_{2n}, Ty\| \cdot \|Ax_{2n}, Sx_{2n}\|) \cdot \|By, Ty\|]^+$$

$$k_2\left[\left(\|Sx_{2n}, Ty\| \cdot \frac{\|Sx_{2n},By\|+\|Ax_{2n},Ty\|}{2}\right) \cdot \|By, Ty\|\right]$$

As limit n$\to \infty$,

$$\|y, By\| \leq k_1[\|y, Ty\| \cdot \|y, y\| \cdot \|By, Ty\|] + k_2\left[\left(\|y, Ty\| \cdot \frac{\|y, By\| + \|y, Ty\|}{2}\right) \cdot \|By, Ty\|\right]$$

$$\leq k_1\alpha max\{\|y, Ty\|, \|y, y\|, \|By, Ty\|\} + k_2\alpha max\left\{\|y, Ty\|, \frac{\|y, By\| + \|y, Ty\|}{2}, \|By, Ty\|\right\} < \|y, By\|$$

and so By = y.

Since, $B(X) \subseteq S(X)$, there exists $w \in X$ such that Sw = y.

If $Aw \neq y$ by (iii) we have,

$$\|Aw, By\| \leq k_1[(\|Sw, Ty\| \cdot \|Aw, Sw\|) \cdot \|By, Ty\|]$$

$$+ k_2\left[\left(\|Sw, Ty\| \cdot \frac{\|Sw, By\| + \|Aw, Ty\|}{2}\right) + \|By, Ty\|\right]$$

$$\therefore \|Aw, y\| \leq k_1[(\|Sw, y\| \cdot \|Aw, Sw\|) \cdot \|y, y\|] + k_2\left[\left(\|Sw, y\| \cdot \frac{\|Sw, y\| + \|Aw, y\|}{2}\right) + \|y, y\|\right]$$

$$\leq k_1\alpha max\{\|Sw, y\|, \|Aw, Sw\|, \|y, y\|\} + k_2\alpha max\left\{\|Sw, y\|, \frac{\|Sw, y\| + \|Aw, y\|}{2}, \|y, y\|\right\} < \|Aw, y\|$$

This implies that Aw = y.

Hence, Aw = Sw = y.

Since A and S are weakly compatible, ASw = SAw and so, if Ay $\neq$By they by (iii) we get

$$\|Ay, y\| = \|Ay, By\|$$
$$\leq k_1[(\|Sy, Ty\| \cdot \|Ay, Sy\|) \cdot \|By, Ty\|]$$
$$+ k_2\left[\left(\|Sy, Ty\| \cdot \frac{\|Sy, By\| + \|Ay, Ty\|}{2}\right) \cdot \|By, Ty\|\right]$$
$$\leq k_1\alpha max\{\|Sy, y\|, \|Ay, Sy\|, \|y, y\|\} + k_2\alpha max\left\{\|Sy, y\|, \frac{\|Sy, y\| + \|Ay, y\|}{2}, \|y, y\|\right\}$$
$$< \|Ay, y\|$$

Hence, $\|Ay, y\| < \|Ay, y\|$ and so Ay = y.

Thus Ay = Sy = By = Ty = y.

That is, y is a common fixed point for A, B, S and T.

The proof is similar when S(X) is assumed to be a closed subset of X.

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