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Prime labeling of Complete Tripartite Graphs $K_{2,m,n}$ and $K_{3,m,n}$.

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ABSTRACT

Let G be a graph with n vertices. Prime labeling of G is the labeling of the vertices using first n positive integers such that labeling of each pair of adjacent vertices are relatively prime. This work is a generalization of [2] which was analogous to [1] in which prime labeling for complete bipartite graphs were considered. In [2] conditions for the existence of prime labeling of complete tripartite graphs of the form $K_{1,m,n}$ was given where $m, n \in \mathbb{Z}^+$. In our work, we have proved that prime labeling of $K_{2,m,n}$ and $K_{3,m,n}$ exists for some values of m and n where $m, n \in \mathbb{Z}^+$. Further, few non-existance cases have been discussed.

Introduction

Graph labeling is one of the active research areas in Graph Theory. Much research can be found in different types of graph labeling. A recent survey on Graph labeling can be found in [3]. The notion of prime labeling was introduced by Roger Entringer in 1980's and was discussed in [5]. Present work on graph labeling is focused on complete bipartite graphs. In our work, which is a generalization of [2], prime labeling for tripartite graph $K_{2,m,n}$ and $K_{3,m,n}$, where $m, n \in \mathbb{Z}^+$ were considered. Some useful definitions to study this work are given below.

Definition 1. A simple graph is *tripartite* if its vertices can be partitioned into three disjoint subsets in such a way that no edge joins two vertices in the same set. Tripartite graph is *complete tripartite* if each vertex in one partite set is adjacent to all the vertices in the other two sets. If the three partite sets have cardinalities l, m and n, then the resulting complete tripartite graph is $K_{l,m,n}$.

Definition 2. Let $G = G(V(G), E(G))$ be a graph where $V(G)$ and $E(G)$ respectively denote the vertex set and the edge set of G. Then, a bijective map $\varphi: V(G) \to \{1, 2, 3, \dots, |V(G)|\}$ is called a *prime labeling* if for each edge uv in $E(G)$, vertex labeling of $\varphi(u)$ and $\varphi(v)$ are relatively prime. A graph which admits prime labeling is known as a prime graph.

Methodology

Here, two Theorems have been proved giving for the existence of prime labeling of graphs of the form $K_{2,m,n}$ and $K_{3,m,n}$. As stated in [1], let $P(t, v)$ be the set of all primes x such that $t < x \le v$.

Theorem 1

(i) Let m, n be positive integers. Then, $K_{2,m,n}$ has prime labeling if and only if $2+m \le \left|P\left(\frac{2+m+n}{2}, 2+m+n\right)\right| + 1$ where $P\left(\frac{2+m+n}{2}, 2+m+n\right)$ denotes the set of all primes p such that $\frac{2+m+n}{2} < p \le 2+m+n$.

(ii) Let m, n be positive integers. Then, $K_{3,m,n}$ has prime labeling if and only if $3+m \le \left|P\left(\frac{3+m+n}{2}, 3+m+n\right)\right| + 1$ where $P\left(\frac{3+m+n}{2}, 3+m+n\right)$ denotes the set of all primes p such that $\frac{3+m+n}{2} < p \le 3+m+n$.

Proof s.(i) First, consider the set of primes given by $X = P\left(\frac{2+m+n}{2}, 2+m+n\right) \setminus \{p_1\}$, where p_1 is a prime. Let $Y = \{1, \cdots, 2+m+n\}$, $Z = \{1, p_1\}$ and $Y' = Y \setminus \{X \cup Z\}$. Further, Z is relatively prime to Y' and X. So, Z and X are relatively prime to Y'. Consider set of $2+m$ points labeling $\{1, \cdots, 2+m\}$. Join each vertex in Z to all vertices in X. Then, join each vertex in Z to all vertices in Y' and then each vertex in X to all vertices in Y'. The resulting graph is tripartite and it is $K_{2,m,n}$ and further it is a prime labeling of $K_{2,m,n}$.

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(iii) Similar proof as in part (i) can be obtained by defining $X = P\left(\frac{3+m+n}{2}, 3+m+n\right) / \{p_1, p_2\}$, where p_1, p_2 are primes, $Y = \{1, \cdots, 3+m+n\}$ and $Z = \{1, p_1, p_2\}$.

Ramanujan primes are used to obtain the next result of our work. It gives a lower bound for possible prime labeling for large n. As mentioned in [1, p.10]; let $\pi(x)$ denote the number of primes less than or equal to x, then the n^{th}*Ramanujan prime* is the least integer R_n for which $\pi(x) - \pi\left(\frac{x}{2}\right) \geq n$ holds for all $x \geq R_n$. The first five Ramanujan primes are $R_1 = 2, R_2 = 11, R_3 = 17, R_4 = 29$ and $R_5 = 41$.

Theorem 2

(i) $K_{2,m,n}$ is prime if $n \geq R_{m+1} - (m+2)$.

(ii) $K_{3,m,n}$ is prime if $n \geq R_{m+2} - (m+3)$.

Proofs. (i) There are at least m primes in the interval $\left(\frac{2+m+n}{2}, 2+m+n\right]$ for $n \geq R_{m+1} - (m+2)$. Denote the first m of these primes by $p_1, p_2, \dots, p_m$, then the sets given by $A_{2,m,n} = \{1,\ p_1\}$, $B_{2,m,n} = \{p_2, p_3, \cdots,\ p_m\}$ and $C_{2,m,n} = \{1, \cdots, 2+m+n\} \setminus (A_{2,m,n} \cup B_{2,m,n})$ give a prime labeling of $K_{2,m,n}$ for $n \geq R_{m+1} - (m+2)$.

(iii) Similar proof as in part (i) can be obtained by defining $A_{3,m,n} = \{1, p_1, p_2\}$, $B_{3,m,n} = \{p_3, p_4, \cdots,\ p_m\}$ and $C_{3,m,n} = \{1, \cdots, 3+m+n\} \setminus (A_{3,m,n} \cup B_{3,m,n})$.

Results and Discussion

In this section, results are proved for $K_{2,m,n}$ when $m = 2$ and 3 and for $K_{3,3,n}$. Also a prime labeling of $K_{2,2,9}$ is illustrated.

Corollary 1

(i) $K_{2,2,n}$ is a prime graph if $n = 9$ **or** $n \geq 13$.

(ii) $K_{2,2,n}$ is not a prime graph if $2 \leq n \leq 12$ excluding 9.

Proofs. (i) (By inspection) If $n = 9$, then $A_{2,2,9} = \{1, 7\}$, $B_{2,2,9} = \{11, 13\}$ and $C_{2,2,9} = \{2, 3, 4, 5, 6, 8, 9, 10, 12\}$ give a prime labeling of $K_{2,2,9}$. The prime graph of $K_{2,2,9}$ is given below:

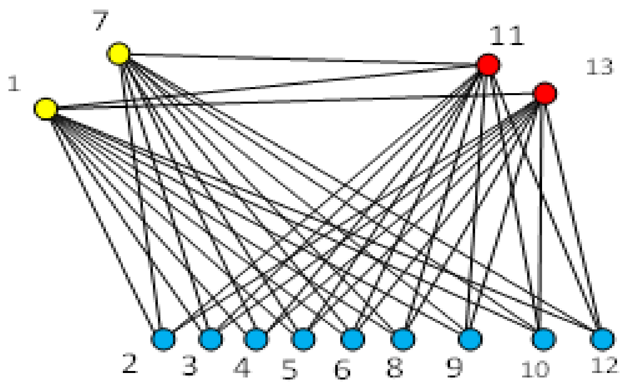

Figure 1. Prime labeling of $K_{2,2,9}$

Since $R_3 = 17$, there are at least three primes p_1, p_2, p_3 in the interval $\left(\left\lfloor\frac{n+4}{2}\right\rfloor, n+4\right]$ for n ≥ 13. Hence, the sets $A_{2,2,n} = \{1, p_1\}$, $B_{2,2,n} = \{p_2, p_3\}$ and $C_{2,2,n} = \{1, \cdots, n+4\} \setminus \left(A_{2,2,n} \cup B_{2,2,n}\right)$ give a prime labeling of $K_{2,2,n}$ for $n \geq 13$.

(iii) When $2 \leq n \leq 12$, excluding 9, we *cannot* obtain two subsets with two vertices in each subset such that the vertices of these subsets are relatively prime with labeling of vertices of the other subset.

Corollary 2

(i) $K_{2,3,n}$ is a prime graph if $n = 14, 15, 16, 18, 19, 20$ **or** $n \geq 24$.

(ii) $K_{2,3,n}$ is not a prime graph if $2 \leq n \leq 13$, $n = 17, 21, 22, 23$.

Proofs. (i) (By inspection), if $n = 14, 15$ or 16 then choose $A_{2,3,n} = \{1, 11\}$ and $B_{2,3,n} = \{13, 17, 19\}$. If $n = 18, 19$ or 20, then choose $A_{2,3,n} = \{1, 13\}$ and $B_{2,3,n} = \{17, 19, 23\}$. In each case, $C_{2,3,n} = \{1, \cdots, n+5\} \setminus (A_{2,3,n} \cup B_{2,3,n})$ gives a prime labeling of $K_{2,3,n}$.

Since $R_4 = 29$, there are at least four primes p_1, p_2, p_3, p_4 in the interval $\left(\left\lfloor\frac{n+5}{2}\right\rfloor, n+5\right]$ for $n \geq 24$. Hence, the sets $A_{2,3,n} = \{1, p_1\}$, $B_{2,3,n} = \{p_2, p_3, p_4\}$ and $C_{2,3,n} = \{1, \cdots, n+5\} \setminus \left(A_{2,3,n} \cup B_{2,3,n}\right)$ give a prime labeling of $K_{2,3,n}$ for $n \geq 24$.

(iii) A similar proof as for Corollary 1.(ii) can be given.

Corollary 3

(i) $K_{3,3,n}$ is a prime graph if $n = 25, 26, 27, 31$ **or** $n \geq 35$.

(ii) $K_{3,3,n}$ is not a prime graph if $2 \leq n \leq 24$, $28 \leq n \leq 34$ except 31.

Proofs. (i) If $n = 25, 26$ or 27 then choose $A_{3,3,n} = \{1, 17, 19\}$ and $B_{3,3,n} = \{23, 29, 31\}$. If $n = 31$, then choose $A_{3,3,n} = \{1, 19, 23\}$ and $B_{3,3,n} = \{29, 31, 37\}$. In each case, $C_{3,3,n} = \{1, \cdots, n+6\} \setminus (A_{3,3,n} \cup B_{3,3,n})$ gives a prime labeling of $K_{3,3,n}$.

Since $R_5 = 41$, there are at least five primes p_1, p_2, p_3, p_4, p_5 in the interval $\left(\left\lfloor \frac{n+6}{2} \right\rfloor, n+6\right]$ for $n \geq 35$. Hence, the sets $A_{3,3,n} = \{1, p_1, p_2\}$, $B_{2,3,n} = \{p_3, p_4, p_5\}$ and $C_{2,3,n} = \{1, \cdots, n+6\} \setminus (A_{3,3,n} \cup B_{3,3,n})$ give a prime labeling of $K_{3,3,n}$ for $n \geq 35$.

(iii) A similar proof as for Corollary 1.(ii) can be given.

The following table gives the prime labeling of tripartite graphs for some m.

Table 1: Prime labeling of $K_{2,m,n}$ and $K_{3,m,n}$ for some values of m and n.

$K_{2,m,n}$	$K_{3,m,n}$	n values for which prime graphs can be obtained
$K_{2,1,n}$		$n = 4, 5, 6$ and $n \geq 8$
$K_{2,2,n}$	$K_{3,1,n}$	$n = 9$ and $n \geq 13$
$K_{2,3,n}$	$K_{3,2,n}$	$n = 14, 15, 16, 18, 19, 20$ and $n \geq 24$
$K_{2,4,n}$	$K_{3,3,n}$	$n = 25, 26, 27, 31$ and $n \geq 35$
$K_{2,5,n}$	$K_{3,4,n}$	$n = 36, 37, 38$ and $n \geq 40$
$K_{2,6,n}$	$K_{3,5,n}$	$n = 45, 46, 47, 48, 49$ and $n \geq 51$
$K_{2,7,n}$	$K_{3,6,n}$	$n = 52$ and $n \geq 58$
$K_{2,8,n}$	$K_{3,7,n}$	$n \geq 61$
$K_{2,9,n}$	$K_{3,8,n}$	$n = 62, 68, 69, 70, 72, 73, 74, 78, 79, 80, 81, 82$ and $n \geq 86$
$K_{2,10,n}$	$K_{3,9,n}$	$n \geq 89$
$K_{2,11,n}$	$K_{3,10,n}$	$n = 90, 91, 92$ and $n \geq 94$

Conclusion

In our work, it has been shown that prime labeling of complete tripartite graphs of the form $K_{2,m,n}$ and $K_{3,m,n}$ exist for some values of m and n where $m, n \in \mathbb{Z}^+$. Those are given in the above table. Also, some non-existing results have been proved.

For a general complete tripartite graph $K_{l,m,n}$; l, m, n can be permuted in $3! = 6$ ways. Moving partite sets with l, m and n number of vertices six graphs can be obtained and all are isomorphic. Hence, permuting $2, m, n$ in $K_{2,m,n}$ and permuting $3, m, n$ in $K_{3,m,n}$ large number of prime graphs can be constructed.

This work can be further generalized for complete bipartite graphs of the form $K_{l,m,n}$ where $l, m, n \in \mathbb{Z}^+$.

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