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The Range of the Hankel type and Extended Hankel type Transformations

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ABSTRACT
In this paper we have studied the range of Hankel type and extended Hankeltype transforms on some spaces of functions. Further the Paley-Wiener type theorem for the Hankel type transforms is also established.

1. Introduction

In view of [27], we define Hankel type transform as

$$f(x) = (\mathcal{H}_{\alpha,\beta}\, g)(x) = \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\, g(y)\, dy, \quad x \in R_+, = (0,\infty) \tag{1.1}$$

if the integral converges in some sense (absolutely, improper, or mean convergence), where $J_{\alpha-\beta}(x)$is the Bessel type function of the first kind [1]. According to [27] if $Re(\alpha-\beta) > -1$, then the Hankel type transform is an automorphism of $L_2(R_+)$ and its inverse on $L_2(R_+)$ has the symmetric form

$$g(x) = (\mathcal{H}_{\alpha,\beta} f)(x) = \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\, f(y)\, dy, x \in R_+, Re(\alpha-\beta) > -1 \cdot \tag{1.2}$$

Following [13,22], we define the extended Hankel type transform as

$$f(x) = (\mathcal{H}_{\alpha,\beta}\, g)(x) = \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta,m}(xy)\, g(y)\, dy, \tag{1.3}$$

$Re(\alpha-\beta) < -1,\ Re(\alpha-\beta) \neq -3,-5,\dots\dots, x \in R_+,\ 1-2m > Re(\alpha-\beta) > -2m-1,\ m > 0$,where$J_{\alpha-\beta,m}(x)$ is the truncated (or "cut") Bessel type function of the first kind and is defined as

$$J_{\alpha-\beta,m}(x) = J_{\alpha-\beta}(x) - \sum_{k=0}^{m-1} \frac{(-1)^k \left(\frac{x}{2}\right)^{\alpha-\beta+2k}}{\Gamma(3\alpha+\beta+k)\, k!}, \tag{1.4}$$

where ,$1-2m > Re(\alpha-\beta) > -2m-1, m \geq 0$

and the integral is understood in L_2 sense. The extended Hankel type transform (1.3) is a bounded operator in $L_2(R_+)$ and its inverse, also a bounded operator in $L_2(R_+)$, has been proved to have the form (see [12])

$$g(x) = -x^{-2\beta} D_x x^{2\beta} \int_0^\infty (xy)^{\alpha+\beta} J_{-\alpha-3\beta,m+1}(xy)\, f(y)\, dy, \tag{1.5}$$

$x \in R_+,\ 1-2m > Re(\alpha-\beta) > -2m-1,\ m > 0,\ D_x \equiv \frac{d}{dx}$.

Formula (1.5) can be rewritten in the equivalent form, symmetric to formula (1.3). In fact, if we put

$$f_N(x) = \begin{cases} f(x), & x \in [1/N,\ N] \\ 0, & otherwise \end{cases} \tag{1.6}$$

Then $f_N(x)$ tends to $f(x)$ in $L_2(R_+)$ norm. Therefore if $g_N(x)$ is the inverse of the extended Hankel type transform (1.5) of $f_N(x)$, then $g_N(x)$tends to $g(x)$ in $L_2(R_+)$norm. By using the relation

$$\frac{d}{dx}\left(x^{\alpha+3\beta} J_{-\alpha-3\beta,m+1}(x)\right) = -x^{\alpha-\beta} J_{\alpha-\beta,m}(x), Re(\alpha-\beta) > -2m-1, m \geq 0, \tag{1.7}$$

we have

$$g_N(x) = -x^{-2\beta} \frac{d}{dx} x^{2\beta} \int_{1/N}^{N} (xy)^{\alpha+\beta} J_{-\alpha-3\beta,m+1}(xy)\, f(y)\, dy$$

$$= \int_{1/N}^{N} (xy)^{\alpha+\beta} J_{\alpha-\beta,m}(xy)\, f(y)\, dy. \tag{1.8}$$

Therefore,

$$g(x) = (\mathcal{H}_{\alpha,\beta} f)(x) = \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta,m}(xy)\, f(y)\, dy, \tag{1.9}$$

$1-2m > Re(\alpha-\beta) > -2m-1, m > 0$, wherethe integral is understood in L_2 sense.

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In [11,21,24] the range of the Hankel transform in the space $\mathcal{L}_p$ with weight has been described through the range of the fractional integral operator and the Fourier cosine transform, or through some Parseval relation. In [28,29] the Hankel transformis proved to be an automorphism of the space of functions $\mathcal{M}_{c,r}^{-1}(L)$ introduced there. In [14] the range of the Hankel transform of infinitely differentiable functions with compact supports has been discussed. Following Zemanian [31], we construct a testing function space $H_{\alpha,\beta}$,$(\alpha-\beta) \in R$,consisting of smooth functions ϕon $(0,\infty)$ such that

$$\rho_{m,k}^{\alpha,\beta}(\phi) = Sup_{0<x<\infty}\left|x^m\left(\frac{1}{x}.\frac{d}{dx}\right)^k\left(x^{2\beta-1}\,\phi(x)\right)\right| < \infty' \tag{1.10}$$

and it can be proved that the Hankel type transform is an automorphism on $H_{\alpha,\beta}$ if $(\alpha-\beta) > -\frac{1}{2}$. The space H_μ studied in Zemanian [31] has been generalized by different ways in [3,4,5,6,7,8,9,10,15,16,17,18,19] to deal with the Hankel transform of distributions.

In the present paper we describe the range of the Hankel type transform on some spaces of functions. One of the main tools in the proofs of our next two theorems is the Plancherel's theorem for the Hankel type transform.

$$\left\|\mathcal{H}_{\alpha,\beta}\, g\right\|_2 = \|g\|_2 \ , \tag{1.11}$$

where,

$\|g\|_p = \|g\|_{L_p(R_+)}$, $1 \le p \le \infty$,that is valid only when $(\alpha-\beta)$ is a real number and $(\alpha-\beta) > -1$. For complex $(\alpha-\beta)$, the Plancherel's equation (1.11) is replaced by the inequalities.

$$C^{-1}\|g\|_2 \le \left\|\mathcal{H}_{\alpha,\beta}g\right\|_2 \le C\,\|g\|_2\,, Re(\alpha-\beta) > -1, \tag{1.12}$$

where$C \in [1,\infty)$ is a constant independent of g. The inequalities (1.12) also holds for the extended Hankel type transform (1.3) for $Re(\alpha-\beta) < -1$, $Re\,(\alpha-\beta) \ne -3,-5\,\ldots$., (See [22]).

2. Hankel type transform of rapid decreasing functions

The range of the Hankel type transform of rapid decreasing and square integrable functions is described by the following :

Theorem 2.1

Let $y^n g(y) \in L_2(R_+)$ for all $n = 0,1,2,\ldots\ldots A$ function $f(x)$ is the Hankel type transform$\mathcal{H}_{\alpha,\beta}$, $Re(\alpha-\beta) \ge 1/2$,of the function $g(y)$ if and only if

(i) $f(x)$is infinitely differentiable on R_+ ;

(ii) $\left[D^2+\frac{1}{x^2}\left(\frac{1}{4}-\alpha^2-\beta^2+2\alpha\beta\right)\right]^n f(x), n = 0,1,\ldots\ldots \in L_2(R_+)$;

(iii) $\left[D^2+\frac{1}{x^2}\left(\frac{1}{4}-\alpha^2-\beta^2+2\alpha\beta\right)\right]^n f(x), n = 0,1,\ldots\ldots tend\ to\ 0\ as\ x \to \quad 0\ and\ to\ infinity;$

(iv) $\frac{d}{dx}\left[D^2+\frac{1}{x^2}\left(\frac{1}{4}-\alpha^2-\beta^2+2\alpha\beta\right)\right]^n f(x),\ n = 0,1,\ldots..tend\ to\ 0\ as\ x \to \infty\ and\ are\ bounded\ at\ 0.$

Proof

First we prove necessary part: Let $y^n g(y) \in L_2(R_+)$for all $n = 0,1,2\ldots\ldots, then\ y^n g(y) \in L_1(R_+)$ for all $n = 0,1,2,\ldots..$ Let $f(x)$ be the Hankel type transform $\mathcal{H}_{\alpha,\beta}$of $g(y)$.

(i) Referring to [1], we have

$$\frac{d^n}{dx^n}J_{\alpha-\beta}(x) = 2^{-n}\sum_{j=0}^{n}(-1)^j\binom{n}{j}J_{\alpha-\beta-n+2j}(x)\,. \tag{2.1}$$

Therefore,

$$\frac{d^n}{dx^n}\left[(xy)^{\alpha+\beta}J_{\alpha-\beta}(xy)\right] = \sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^{n+j-k}2^{-k}\left(-\frac{1}{2}\right)_{n-k}\binom{n}{k}\binom{k}{j}\times\ x^{\alpha+\beta+k-n}y^{\alpha+\beta+k}J_{\alpha-\beta-k+2j}(xy) \tag{2.2}$$

Here $(a)_n$ is the Pochhammer symbol defined by $(a)_n = \Gamma(a+n)/\Gamma(a)$.

The Bessel type function of the first kind $J_{\alpha-\beta}(y)$has the asymptotics [1]

$$J_{\alpha-\beta}(y) = \begin{cases}\sqrt{\dfrac{2}{\pi y}}\left[Cos\left(y-\dfrac{(\alpha-\beta)\pi}{2}-\dfrac{\pi}{4}\right)+\dfrac{1-4(\alpha-\beta)^2}{8y}\sin\left(y-(\alpha-\beta)\dfrac{\pi}{2}-\dfrac{\pi}{4}\right) + O(y^{-2})\right], & y\to\infty,\\ O\left(y^{Re(\alpha-\beta)}\right) & ,y\to 0\end{cases}$$

Consequently,

$$\frac{\partial^n}{\partial x^n}\left[(xy)^{\alpha+\beta}J_{\alpha-\beta}(xy)\right] \tag{2.3}$$

as a function of y has the order $O\left(y^{\alpha+\beta+Re(\alpha-\beta)}\right)$ in the neighbourhood of 0 and $O(y^n)$at infinity. Thus $\frac{\partial^n}{\partial x^n}\left[(xy)^{\alpha+\beta}J_{\alpha-\beta}(xy)\right]g(y)$, $Re(\alpha-\beta) > -1$, as a function of y belongs to $L_1(R_+)$ for all $n = 0,1,2,\ldots..$ Thus $f(x)$ is infinitely differentiable on R_+.

(ii) Since $J_{\alpha-\beta}(x)$satisfies the differential equation [1]

$$x^2u'' + xu' + (x^2-\alpha^2-\beta^2+2\alpha\beta)u = 0 \tag{2.4}$$

Then $x^{\alpha+\beta}J_{\alpha-\beta}(x)$ is a solution of the equation

$$x^2u'' + \left(x^2-\frac{1}{4}-\alpha^2-\beta^2+2\alpha\beta\right)u = 0\cdot \tag{2.5}$$

Therefore, we have

$$\left[\frac{\partial^2}{\partial x^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^n \left((xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\right) = (-y^2)^n (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy). \quad (2.6)$$

Consequently,

$$\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^n f(x) = (-1)^n \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\, y^{2n}\, g(y)\, dy, \qquad Re\,(\alpha-\beta) - 1\ . \quad (2.7)$$

Because of the Plancherel's inequality (1.12) and $y^{2n}\, g(y) \in L_2(R_+)$,

we obtain that $\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^n f(x), Re\,(\alpha-\beta) > -1,\ n = 0,1,2\ldots . \in L_2(R_+)$.

(iii) For the kernel $(xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)$ has the asymptotics $x^{2\alpha}$ as $x \to 0$, is uniformly bounded on $(0,\infty)$ if $Re(\alpha-\beta) \geq -\frac{1}{2}$, and $y^{2n} g(y) \in L_1(0,\infty)$, then applying the dominated convergence theorem, we have

$$\lim_{x\to 0}\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^n f(x)$$

$$= (-1)^n \int_0^\infty \lim_{n\to\infty}[(xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)]\, y^{2n} g(y)\, dy = 0\ ,\ Re(\alpha-\beta) > -\frac{1}{2}. \quad (2.8)$$

Following the same argument for $\epsilon > 0$, we can choose N large enough so that

$$\left|\int_N^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\, y^{2n}\, g(y)\, dy\right| < \epsilon, \quad (2.9)$$

uniformly with respect to $x \in R_+$. The Bessel type function $J_{\alpha-\beta}(y)$ has the asymptotics (2.3), therefore the integral

$$\int_{ax}^{bx} y^{\alpha+\beta} J_{\alpha-\beta}(y)\, dy\ ,\ Re(\alpha-\beta) \geq -\frac{1}{2}, \quad (2.10)$$

is uniformly bounded for all non-negative a, b and x. Hence

$$\int_a^b (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\, dy = \frac{1}{x}\int_{ax}^{bx} y^{\alpha+\beta} J_{\alpha-\beta}(y)\, dy,\ \ Re(\alpha-\beta) \geq -\frac{1}{2}, \quad (2.11)$$

tends to 0 uniformly in a, b for $0 \leq a < b < \infty$ as $x \to \infty$. Now by applying the generalized Riemann-Lebesgue theorem [27], we obtain

$$\lim_{x\to\infty} \int_0^N (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\, y^{2n}\, g(y)\, dy = 0\ , 0 < N < \infty,\ \ Re(\alpha-\beta) \geq -\frac{1}{2}. \quad (2.12)$$

Since ϵ can be taken arbitrarily small, from (2.9) and (2.12), we have

$$\lim_{x\to\infty} \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\, y^{2n}\, g(y)\, dy = 0,\ \ Re(\alpha-\beta) \geq -\frac{1}{2}. \quad (2.13)$$

Thus,

$$\lim_{x\to\infty}\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^n f(x) = 0, n = 0,1,\ldots, Re(\alpha-\beta) \geq -\frac{1}{2} \quad (2.14)$$

(iv) Using the formula [1]

$$\frac{\partial}{\partial x}[(xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)] = \frac{1}{2}\left(\frac{y}{x}\right)^{\alpha+\beta} J_{\alpha-\beta}(xy) + \frac{y}{2}(xy)^{\alpha+\beta} \times [J_{-\alpha-3\beta}(xy) - J_{3\alpha+\beta}(xy)] \quad (2.15)$$

we have

$$(-1)^n \frac{d}{dx}\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^n f(x)$$

$$= \frac{1}{2x}\int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\, y^{2n}\, g(y)\, dy + \frac{1}{2}\int_0^\infty (xy)^{\alpha+\beta} J_{-\alpha-3\beta}(xy)\, y^{2n+1}\, g(y)\, dy$$

$$-\frac{1}{2}\int_0^\infty (xy)^{\alpha+\beta} J_{3\alpha+\beta}(xy)\, y^{2n+1}\, g(y)\, dy\ . \quad (2.16)$$

Now from (iii) and especially from (2.8) and (2.13) we see that when $Re(\alpha-\beta) \geq \frac{1}{2}$ the first and the second expressions on the right hand side of (2.16) as functions of x tend to 0 at infinity, whereas the third expression tends to 0 both at 0 and infinity. As $(xy)^{-(\alpha+\beta)} J_{\alpha-\beta}(xy)$ and $(xy)^{\alpha+\beta} J_{-\alpha-3\beta}(xy),\ Re(\alpha-\beta) \geq 1/2$ are uniformly bounded, the first and second expressions on the right hand side of (2.16) are uniformly bounded on R_+ and in particular at 0.Hence

$$\frac{d}{dx}\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^n f(x), \qquad Re\,(\alpha-\beta) \geq 1/2$$

tends to 0 at infinity and is bounded at 0.

Now we prove sufficiency part: Suppose that f satisfies the conditions (i) to (iv) of the theorem. Then $\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^n f(x) \in L_2(R_+)$ for all

$n = 0,1,2, \ldots\ldots$ Let $g_n(y)$ be its Hankel type transform, that means

$$g_n(y) = \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy) \left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^n f(x)\, dx, \quad (2.17)$$

$Re(\alpha-\beta) \geq 1/2\ ,\ n = 0,1,2, \ldots\ldots$.

where the integral is understood in L_2 sense. Putting

$$g_n^N(y) = \int_{1/N}^N (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy) \left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^n f(x)\, dx, \quad (2.18)$$

$n = 0,1,2, \ldots\ldots$,

We see that $g_n^N(y)$ tends to $g_n(y)$ in L_2 norm as $N \to \infty$. Let $n \geq 1$.Integrating (2.18) by parts twice we obtain

$$g_n^N(y) = \left\{(xy)^{\alpha+\beta} J_{\alpha-\beta}(xy) \frac{d}{dx}\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^{n-1} f(x)\right\}\Big|_{x=1/N}^{N}$$
$$-\left\{\frac{\partial}{\partial x}\left((xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\right)\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^{n-1} f(x)\right\}\Big|_{x=1/N}^{N}$$
$$+ \int_{1/N}^{N}\left[\frac{\partial^2}{\partial x^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]\left((xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\right)\times \left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^{n-1} f(x)\,dx. \quad (2.19)$$

Now using formulas (2.6) and (2.15) we get

$$g_n^N(y) = (Ny)^{\alpha+\beta} J_{\alpha-\beta}(Ny) \frac{d}{dx}\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^{n-1} f(N) \quad (2.20)$$
$$-\left(\frac{y}{N}\right)^{\alpha+\beta} J_{\alpha-\beta}(y/N) \frac{d}{dx}\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^{n-1} f(1/N) \quad (2.21)$$
$$-\frac{1}{2}\left(\frac{y}{N}\right)^{\alpha+\beta} J_{\alpha-\beta}(Ny)\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^{n-1} f(N) \quad (2.22)$$
$$-\frac{1}{2}\, y\,(Ny)^{\alpha+\beta} J_{-\alpha-3\beta}(Ny)\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^{n-1} f(N) \quad (2.23)$$
$$+\frac{1}{2}\, y\,(Ny)^{\alpha+\beta} J_{3\alpha+\beta}(Ny)\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^{n-1} f(N) \quad (2.24)$$
$$+\frac{1}{2}(Ny)^{\alpha+\beta} J_{\alpha-\beta}(y/N)\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^{n-1} f(1/N) \quad (2.25)$$
$$+\frac{1}{2}\, y\left(\frac{y}{N}\right)^{\alpha+\beta} J_{-\alpha-3\beta}(y/N)\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^{n-1} f(1/N)) \quad (2.26)$$
$$-\frac{1}{2}\, y\left(\frac{y}{N}\right)^{\alpha+\beta} J_{3\alpha+\beta}(y/N)\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^{n-1} f(1/N) \quad (2.27)$$
$$-y^2\int_{1/N}^{N}(xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^{n-1} f(x)\,dx \quad (2.28)$$

Here $P\left(\frac{d}{dx}\right) f(N)$ means $P\left(\frac{d}{dx}\right) f(x)|_{x=N}$.

Since $(Ny)^{\alpha+\beta} J_{\alpha-\beta}(Ny)$ is uniformly bounded and $\frac{d}{dx}\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^{n} f(N)$ tends to 0 as $N \to \infty$.(property (iv)), the expression on the right hand side of (2.20)tends to 0 as $N \to \infty$.

Using property (iv) we see that $\frac{d}{dx}\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^{n-1} f(1/N)$ is bounded, whereas function $(y/N)^{\alpha+\beta} J_{\alpha-\beta}(y/N)$ has the order $O(N^{2\beta-1})$ at infinity. Hence, expression (2.21) tends to 0 as $N \to \infty$.

Similarly, function $\frac{1}{2}(y/N)^{\alpha+\beta} J_{\alpha-\beta}(Ny)$ has the order $O(N^{-1})$ and $\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^{n-1} f(N)$ is $O(1)(property\ (iii))$, therefore both expressions (2.23) and (2.24) tend to 0 as $N \to \infty$.

Functions $\frac{1}{2}(Ny)^{\alpha+\beta} J_{\alpha-\beta}(y/N)$, $\frac{1}{2}\, y\,(y/N)^{\alpha+\beta} J_{-\alpha-3\beta}(y/N)$, and $\frac{1}{2}\, y\,(y/N)^{\alpha+\beta} J_{3\alpha+\beta}(y/N)$ are bounded, hence all the expressions (2.25), (2.26) and (2.27) tend to 0 as $N \to \infty$.

Function (2.28) converges to $-y^2 g_{n-1}(y)$ as N approaches infinity, hence $g_n(y) = -y^2 g_{n-1}(y)$, and therefore $g_n(y) = (-y^2)^n g_0(y)$, $n = 0,1,\dots\dots$

But if g is the Hankel type transform (1.2) of f, then f is the Hankel type transform (1.1)of g. Therefore, $f(x)$ is the Hankel type transform of a function $g(y) = g_0(y)$ such that $y^{2n} g(y) \in L_2(R_+)$, $n = 0,1,\dots\dots$ and Theorem 2.1 is proved.

Corollary 2.1

The Zemanian type space $\mathcal{H}_{\alpha,\beta}$, $Re\,(\alpha-\beta) \geq 1/2$ can be characterized as the set of functions $f(x)$, satisfying conditions (i) to (iv) of Theorem 2.1 and such that $x^n f(x) \in L_2(R_+)$, $n = 0,1,2 \dots\dots$

Proof

It is well known [5] that $f \in H_{\alpha,\beta}$ if and only if $f(x) = x^{2\alpha}\phi(x)$, where $\phi \in S_{even}$, the set of restrictions of even Schwartz functions on R_+. It is proved [8] that $\phi \in S_{even}$ if and only if

$$Sup_{x\in R_+}|x^n\phi(x)| < \infty,\quad Sup_{x\in R_+}|x^n\psi(x)| < \infty,\quad n = 0,1,\dots\dots, \quad (2.29)$$

where

$$\psi(x) = \int_0^\infty (xy)^{-(\alpha-\beta)} J_{\alpha-\beta}(xy)\,\phi(y) y^{4\alpha}\,dy. \quad (2.30)$$

Thus $f \in H_{\alpha,\beta}$ if and only if

$$Sup_{x\in R_+}\left|x^{n+2\beta-1} f(x)\right| < \infty,\ \sup_{x\in R_+}\left|x^n \mathcal{H}_{\alpha,\beta} f(x)\right| < \infty, n = 0,1,\dots. \quad (2.31)$$

Suppose that f satisfies conditions (i) to (iv) of Theorem 2.1 and $x^n f(x) \in L_2(R_+)$, $n = 0,1,2 \dots\dots$ Then $x^2\beta - 1$ is bounded at 0.Since $x^n f(x) \to 0$ at 0 and infinity, $x^{n+2\beta} f(x)$ is bounded on R_+.Since the Hankel type transform has the symmetric inverse, $x^{n+2\beta-1}\mathcal{H}_{\alpha,\beta} f(x)$ is also bounded on R_+. But it is equivalent to the fact that $f \in H_{\alpha,\beta}$.

Now suppose that $f \in H_{\alpha,\beta}$. Then $\mathcal{H}_{\alpha,\beta} f \in H_{\alpha,\beta}$. Hence, inequality (1.10) is valid with $k = 0$ for both f and $\mathcal{H}_{\alpha,\beta} f$:

$$\left|x^{n+2\beta-1} f(x)\right| < \infty,\ \left|x^{n+2\beta-1}\mathcal{H}_{\alpha,\beta} f(x)\right| < \infty,\ n = 0,1,\dots\dots \quad (2.32)$$

Hence, f satisfies conditions (i) to (iv) of Theorem 2.1 and

$x^n f(x) \in L_2(R_+), \quad n = 1,2 \dots ..$

Thus proof is completed.

3. Hankel type transform of square integrable functions with compact supports

Here we discuss the Hankel type transforms of square integrable functions with compact supports.

Theorem 3.1: (The Paley-Wiener Theorem)

A function f is the Hankel type transform $\mathcal{H}_{\alpha,\beta}$, $Re(\alpha-\beta) \geq 1/2$, of a square integrable function g with compact support on $[0,\infty)$ if and only if f satisfies conditions (i) to (iv) of Theorem 2.1 and moreover,

$$\lim_{n\to\infty} \left\| \left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^n f(x) \right\|_2^{\frac{1}{2n}} = \sigma_g < \infty, \quad (3.1)$$

Where $\sigma_g = Sup\,\{y : y \in Supp\ g\}$, (3.2)

and the support of a function is the smallest closed set, outside it the function vanishes almost everywhere [30].

Proof

(a) Let $f(x)$ be the Hankel type transform of $g(y) \in L_2(R_+)$ and $\sigma_g < \infty$:

$$f(x) = \int_0^{\sigma_g} (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\, g(y)\, dy, \ Re(\alpha-\beta) \geq 1/2. \quad (3.3)$$

One can assume that $\sigma_g > 0$, otherwise it is trivial. Since $\sigma_g < \infty$, we have $y^n g(y) \in L_2(R_+)$ for all $n = 0,1,2, \dots ..$ Therefore, f satisfies conditions (i) to (iv) of Theorem 2.1.

Furthermore,

$$\left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^n f(x) = \int_0^{\sigma_g} (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)(-y^2)^n\, g(y)\, dy. \quad (3.4)$$

By applying the right hand side inequality in (1.12) for the Hankeltype transform (3.4), we have

$$\left\| \left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^n f(x) \right\|_2^2 \leq C \int_0^{\sigma_g} y^{4n} |g(y)|^2\, dy$$

$$\leq C\, \sigma_g^{4n} \int_0^{\sigma_g} |g(y)|^2\, dy. \quad (3.5)$$

Hence

$$\overline{\lim_{n\to\infty}} \left\| \left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^n f(x) \right\|_2^{1/(2n)} \leq \overline{\lim_{n\to\infty}}\, C^{1/(4n)} \sigma_g \left\{\int_0^{\sigma_g} |g(y)|^2\, dy\right\}^{1/4n} = \sigma_g. \quad (3.6)$$

On the other hand, as σ_g is the least upper bound of the support of g, for every ϵ , $0 < \epsilon < \sigma_g$, we have

$$\int_{\sigma_g-\epsilon}^{\sigma_g} |g(y)|^2\, dy > 0. \quad (3.7)$$

Consequently, using the left hand side inequality in (1.12), we have

$$\overline{\lim_{n\to\infty}} \left\| \left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^n f(x) \right\|_2^{1/(2n)} \geq \frac{\lim}{n\to\infty} C^{-1/(4n)} \left\{ \int_{\sigma_g-\epsilon}^{\sigma_g} y^{4n} |g(y)|^2\, dy \right\}^{1/4n}$$

$$\geq (\sigma_g - \epsilon) \lim_{n\to\infty} C^{-1/(4n)} \left\{\int_{\sigma_g-\epsilon}^{\sigma_g} |g(y)|^2\, dy\right\}^{1/4n} = \sigma_g - \epsilon. \quad (3.8)$$

Because ϵ can be chosen arbitrarily small, from (3.8) and (3.6) we obtain (3.1).

(b) Suppose that f satisfies the conditions (i) to (iv) of Theorem 2.1, and the limit in (3.1) existsand equals $\sigma < \infty$. Using Theorem 2.1 we see that f is the Hankel type transform of a function g such that $y^n g(y) \in L_2(R_+)$, $n = 0,1,2 \dots ..$ We shall prove that $\sigma_g < \infty$, and moreover, $\sigma = \sigma_g$. From Theorem 2.1we have that (2.7) is valid. Therefore by applying the inequalities (1.12) we obtain

$$C^{-1} \|y^{2n} g(y)\|_2 \leq \left\| \left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^n f(x) \right\|_2 \leq C\, \|y^{2n} g(y)\|_2 . \quad (3.9)$$

Thus

$$\lim_{n\to\infty} C^{-1/(2n)} \|y^{2n} g(y)\|_2^{1/(2n)} \leq \lim_{n\to\infty} \left\| \left[\frac{d^2}{dx^2} + \frac{1}{x^2}\left(\frac{1}{4} - \alpha^2 - \beta^2 + 2\alpha\beta\right)\right]^n f(x) \right\|_2^{1/(2n)}$$

$$= \sigma \leq \lim_{n\to\infty} C^{1/(2n)} \|y^{2n}\, g(y)\|_2^{1(2n)}. \quad (3.10)$$

Consequently,

$$\lim_{n\to\infty} \|y^{2n}\, g(y)\|_2^{1/(2n)} = \sigma . \quad (3.11)$$

Suppose that $\sigma_g > \sigma$.Then there exists a positive ϵ such that

$$\int_{\sigma+\epsilon}^{\infty} |g(y)|^2\, dy > 0. \quad (3.12)$$

We have

$$\sigma = \lim_{n\to\infty} \|y^{2n}\, g(y)\|_2^{1/(2n)} \geq \frac{lim}{n\to\infty} \left\{ \int_{\sigma+\epsilon}^{\infty} y^{4n} |g(y)|^2\, dy \right\}^{1/(4n)}$$

$$\geq (\sigma+\epsilon) \lim_{n\to\infty} \left\{\int_{\sigma+\epsilon}^{\infty} y^{4n} |g(y)|^2\, dy\right\}^{1/(4n)} = \sigma + \epsilon, \quad (3.13)$$

that is impossible. Hence $\sigma_g \leq \sigma$ and therefore, g has a compact support. Suppose that $\sigma_g \leq \sigma$. Then there exists $\epsilon > 0$ such that

$$\int_{\sigma-\epsilon}^{\infty} |g(y)|^2\, dy = 0. \quad (3.14)$$

We have

$$\sigma = \lim_{n\to\infty} \|y^{2n}\, g(y)\|_2^{1/(2n)} \leq \overline{\lim_{n\to\infty}} \left\{\int_0^{\sigma-\epsilon} y^{4n} |g(y)|^2\, dy\right\}^{1/(4n)}$$

$$\leq (\sigma-\epsilon)\, \overline{\lim_{n\to\infty}} \{\int_0^{\sigma-\epsilon} |g(y)|^2\, dy\}^{1/(4n)} = \sigma - \epsilon, \quad (3.15)$$

that is impossible. Hence $\sigma_g \geq \sigma$, and consequently $\sigma_g = \sigma < \infty$.

Thus Theorem 3.1 is proved.

Remark 3.1

From Theorem 2.1 and Theorem 3.1 it is not difficult to see that if a function f satisfies conditions (i) to (iv) of Theorem 2.1, then the limit (3.1) always exists. It equals infinity, if the Hankel type transform f has an unbounded support.

4. Hankel type transform of infinitely differentiable functions with compact supports

Let the Erdelyi-Kober fractional type integral operator $\mathcal{K}^{\alpha-\beta}$ be defined by

$$h_1(x) = (\mathcal{K}^{\alpha-\beta} g_1)(x) = \int_x^{\infty} \frac{(y^2-x^2)^{-\alpha-3\beta}}{\Gamma(\alpha-\beta)}\, y\, g_1(y)\, dy, \;\; Re(\alpha-\beta) > 0, \;\; x \in R \quad (4.1)$$

Now we need the following :

Lemma 4.1

The restriction of the Erdelyi-Koberfractional type integral operator $\mathcal{K}^{\alpha-\beta}$in (4.1) on R_+ is a bijection on $S_{even}$and a bijection on its subspace of functions with compact supports. Moreover, $\sigma_{h_1} = \sigma_{g_1}$.

Proof

Following [25], it can be proved that the Weylfractional type integral operator

$$v(x) = (\mathcal{L}_-^{\alpha-\beta} u)(x) = \int_x^{\infty} \frac{(y-x)^{-\alpha-3\beta}}{\Gamma(\alpha-\beta)}\, u(y)\, dy, \;\; x \in R, \quad (4.2)$$

is a bijection in the space of infinitely differentiable functions which have for $x \to \infty$ the same behavior as functions in $S(R)$ and have a slow growth for $x \to -\infty$. From [14], it is known that the Weyl fractional type integral operator (4.2) is also a bijection on the space of infinitely differentiable functions with compact supports on $\bar{R}_+$. From (4.2), it is easy to see that $\sigma_v \leq \sigma_u$.Since the inverse of the Weyl fractional type operator has the form [25]

$$u(x) = (-1)^m \frac{d^m}{dx^m} (\mathcal{L}_-^{m-\alpha+\beta}\, v)(x) = (-1)^m \frac{d^m}{dx^m} \int_x^{\infty} \frac{(y-x)^{m-(3\alpha+\beta)}}{\Gamma(m-\alpha+\beta)}\, v(y)\, dy, \quad (4.3)$$

$m > Re(\alpha-\beta)$,one can reverse the inequality $\sigma_u \leq \sigma_v$. Hence, $\sigma_u = \sigma_v$.

On $\bar{R}_+$ formula (4.3) can be rewritten in the form (4.1) with $2u(y^2) = g_1(y)$, $v(x^2) = h_1(x)$. One can prove that $u, v \in S(R)$ if and only if $g_1, h_1 \in S_{even}$ and u, v have compact supports if and only if g_1, h_1 have compact supports, and moreover,$\sigma_u^2 = \sigma_{g_1}, \sigma_v^2 = \sigma_{h_1}$ .Hence if $\sigma_u = \sigma_v$, then $\sigma_{g_1} = \sigma_{h_1}$and vice versa. Thus Lemma 4.1 is proved.

Theorem 4.1

(The Paley-Wiener-Schwartz type theorem). A function $f \in H_{\alpha,\beta}$is the Hankel type transform $\mathcal{H}_{\alpha,\beta}$, $Re(\alpha-\beta) > -1/2$, of a function $g \in H_{\alpha,\beta}$with compact support if and only if

$$\sigma_g = \lim_{n\to\infty} \left\| \frac{d^n}{dx^n} x^{2\beta-1} f(x) \right\|_p^{1/n}, \;\; 1 \leq p \leq \infty \quad (4.4)$$

Proof

The Bessel type function $J_{\alpha-\beta}(x)$ has the integral representation [1]

$$J_{\alpha-\beta}(x) = \frac{2^{\alpha+3\beta} x^{\alpha-\beta}}{\sqrt{\pi}\Gamma(2\alpha)} \int_0^1 (1-t^2)^{-2\beta}\, Cos\, xt\, dt, \;\; Re(\alpha-\beta) > -1/2\,. \quad (4.5)$$

Substituting x by xy and t by t/ywe have

$$J_{\alpha-\beta}(xy) = \frac{2^{\alpha+3\beta} x^{\alpha-\beta} y^{-(\alpha-\beta)}}{\sqrt{\pi}\Gamma(2\alpha)} \int_0^y (y^2-t^2)^{-2\beta}\, Cos\, xt\, dt\,. \quad (4.6)$$

Consequently, the Hankel type transform (1.1) can be rewrittenin the form

$$f(x) = \frac{2^{\alpha+3\beta} x^{2\alpha}}{\sqrt{\pi}\Gamma(2\alpha)} \int_0^{\infty} y^{2\beta}\, g(y) \int_0^y (y^2-t^2)^{-2\beta}\, Cos\, xt\, dt\, dy. \quad (4.7)$$

If $y^{2\alpha} g(y) \in L_1(R_+)$then the repeated integral (4.7) converges absolutely. Therefore, one can apply the Fubini-Tonelli theorem [30] to interchange the order of integration in (4.7) to get

$$f(x) = \frac{2^{\alpha+3\beta} x^{2\alpha}}{\sqrt{\pi}\Gamma(2\alpha)} \int_0^{\infty} \cos xt \int_t^{\infty} (y^2-t^2)^{-2\beta} y^{2\beta} g(y)\, dy\, dt \quad (4.8)$$

Putting $y^{2\beta-1} g(y) = g_1(y)andx^{2\beta-1} f(x) = f_1(x)$, we have

$$f_1(x) = \frac{2^{\alpha+3\beta}}{\sqrt{\pi}\Gamma(2\alpha)} \int_0^{\infty} \cos xt \int_t^{\infty} (y^2-t^2)^{-2\beta}\, y\, g_1(y)\, dy\, dt\,. \quad (4.9)$$

Therefore, f_1 can be viewed as a composition of the Fourier Cosine transform

$$f_1(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} Cos\, xt\; h_1(t)\, dt, \;\; 0 \leq x < \infty' \quad (4.10)$$

andthe Erdelyi-Kober fractional type integral operator $\mathcal{K}^{2\alpha}$in (4.1) of order 2α, multiplied by a constant. From [2] we see that $\hat{f} \in S(R)$ is the Fourier transform of an infinitely differentiable function f on R with compact support if and only if

$$\sigma_{|f|} = \lim_{n\to\infty} \left\| \frac{d^n}{dx^n} \hat{f}(x) \right\|_{L_p(R)}^{1/n}, \quad 1 \le p \le \infty, \tag{4.11}$$

where

$$\sigma_{|f|} = Sup\,\{|y| : y \in supp\ f\}. \tag{4.12}$$

Restricting the Fourier transform only on even functions we see that a function$f_1 \in S_{even}$is the Fourier Cosine transform (4.10) of a function $h_1 \in S_{even}$with compact support if and only if

$$\sigma_{h_1} = \lim_{n\to\infty} \left\| \frac{d^n}{dx^n} f_1(x) \right\|_p^{1/n}. \tag{4.13}$$

On the other hand, the Erdelyi-Kober fractional integral type operator $\mathcal{K}^{2\alpha}$in (4.1) is a bijection in the space of infinitely differentiable functions on $\bar{R}_+$ with compact supports and $\sigma_{h_1} = \sigma_{g_1}$(Lemma 4.1).
Remarking that if $g_1(y) = y^{2\beta-1}\, g(y)$then $\sigma_{g_1} = \sigma_g$, Theorem 4.1 follows now from formula (4.9).

Remark 4.1

Theorem 3.1 and 4.1 involve the spectral radii [30] of some differential operators (see formulas (3.1) and (4.4)), but the proofs are obtained straight forward, without referring to the spectral theory. In [14] the Hankel type transform of infinitely differentiable functions with compact supports has been described by classical way through entire functions of exponential type.

5. Hankel type and extended Hankel type transform of analytic functions

Let $\mathcal{H}_G$ be the space of functions $g(z)$ that are
(i) regular in the range $-a < org\ z < b$, where $0 < a, b \le \pi$;
(ii) of the order $O(|z|^{-p-\epsilon})$for small z, and $O(|z|^{-q+\epsilon})$ forlarge z,where $p < \frac{1}{2} < q$,uniformly in any angle interior to the above for any small positive ϵ
(iii) Satisfying the condition

$$\int_0^\infty y^{2\alpha+2n}\, g(y)\, dy = 0, \tag{5.1}$$

for all non-negative integers on the intervals

$$(p/2 - Re\,(\alpha-\beta)/2) - 1/4\,, q/2 - Re\left((\alpha-\beta)/2 - \frac{1}{4}\right),$$ provided there exists one.

Let $\mathcal{H}_F$be Theorem 4.1. The Hankel type transform (1.1) and the extended Hankel type transform (1.3) map, one-to-one onto the space of functions $f(z)$, regular in the angle $-b < org\ z < a$, of the order $O(|z|^{1-q-\epsilon})$ for small z., and $O(|z|^{1-p+\epsilon})$ for large z,uniformly in any angle interior to the above for any small positiveϵ, and satisfying conditions

$$\int_0^\infty x^{2\alpha+2n} f(x)\, dx = 0, \tag{5.2}$$

for all non-negative integers n on the interval

$$-q/2 - Re((\alpha-\beta)/2) - \frac{1}{4}\,, -p/2 - Re\left((\alpha-\beta/2) - \frac{1}{4}\right).$$

Proof

Let $g(z)$ satisfy the conditions of Theorem 5.1. Then function $g(z)$ on R_+ belongs to $L_2(R_+)$and its Mellin transform $g^*(s)$ defined by [27]

$$g^*(s) = \int_0^\infty x^{s-1}\, g(x)\, dx\,, \quad Res = \frac{1}{2}\,, \tag{5.3}$$

is an analytic function of s, regular for $a < Res < b$, and

$$g^*(s) = \begin{cases} O\left(e^{-(b-\epsilon)Ims}\right), Ims \to \infty \\ O\left(e^{-(a-\epsilon)Ims}\right), Ims \to -\infty \end{cases} \tag{5.4}$$

for every positive ϵ, uniformly in any strip interior to $p < Res < q$ (see [27]).Let further, $f(x)$ be the Hankel type transform (1.1) or the extended Hankel type transform (1.3) of $g(y)$and $f^*(s)$ be its Mellin transform (5.3). Since $g(y) \in L_2(R_+)$the Hankel type and extended Hankel type transforms in $L_2(R_+)$are equivalent to the following Parseval equation for $f^*(s)$ and $g^*(s)$ in$L_2(1/2 - i\infty,\ 1/2 + i\infty)$on the line $Res = 1/2$ [27,13] :

$$f^*(s) = 2^{s-\frac{1}{2}} \frac{\Gamma(\alpha+s/2)}{\Gamma(\alpha-s/2)} g^*(1-s)\cdot \tag{5.5}$$

Because of (5.1) the function $g^*(1-s)$ equals 0 at the poles of the gamma function $\Gamma(\alpha+s/2)$ in the strip $1-q < Res < 1-p$, if there exists one. Hence from (5.5), we can see that $f^*(s)$ is analytic in the strip $1-q < Res < 1-p$.Furthermore since the function

$$\frac{2^{s-1/2}\,\Gamma(\alpha+s/2)}{\Gamma(2\alpha+\beta-s/2)}$$

Is uniformly bounded on any compact domain in the strip $1-q < Res < 1-p$, containing no poles of the function $\Gamma(\alpha+s/2)$, and has at most polynomial growth at infinity, from (5.4) we see that the function $f^*(s)$also decays exponentially

$$f^*(s) = \begin{cases} O\left(e^{(b-\epsilon)\,Ims}\right),\ Ims \to -\infty \\ O\left(e^{-(a-\epsilon)\,Ims}\right), Ims \to \infty \end{cases}, \tag{5.6}$$

for every positive ϵ, uniformly in any strip interior to $(1-q) < Res < (1-p)$. Hence its inverse Mellin transform $f(z)$ is regular for $-b < \arg\ z < a$, and of the order $O(|z|^{q-1-\epsilon})$for small z, and $O(|z|^{q-1+\epsilon})$for large z, uniformly in any angle interior to the above angle [27]. Moreover, $f^*(s)$ has zeros at poles of the gamma function $\Gamma(2\alpha+\beta-s/2)$ in the strip $1-q < Res < (1-p)$, if there exists one, therefore (5.2) is satisfied.

Conversely, let $f(z)$ satisfy the conditions of Theorem 4.1. Then $f(x) \in L_2(R_+)$ and its Mellin transform $f^*(s)$ is analytic in the strip $1-q < Res < 1-p$ has zeros at poles of gamma function $\Gamma(2\alpha+\beta-s/2)$ in the strip $1-q < Res < 1-p$ if there exists one, and satisfies (5.6). Thus if we express $f^*(s)$ in the form (5.5), function $g^*(s)$ is analytic in the strip $p < Res < q$; and has the exponential decay (5.4) for every positive ϵ ,uniformly in any strip interior to $p < Res < q$. Furthermore $g^*(1-s)$ has zeros at poles of the gamma function $\Gamma(\alpha+s/2)$ in the strip $p < Res < q$, hence formula (5.1) is valid for all non-negative integers n on the interval $(a/2 - Re\,(\alpha-\beta)/2 - 1/4\ ,\ \ b/2 - Re\,(\alpha-\beta)/2 - 1/4)$, if there exists one. Thus Theorem 5.1 is proved.

If in Theorem 5.1 we take $b = a$ and $\frac{1}{2} - p = q - \frac{1}{2}$, then the spaces of functions f and g coincide, hence denoting $1/2 - p$ again by p we have:

Corollary 5.1

The Hankel type transform (1.1) and the extended Hankel type transform (1.3) are the bijections in the space of functions $g(z)$, regular in the angle $|\arg z| < a$, where $0 < a \le \pi$; of the order $O(|z|^{-1/2+p-\epsilon})$ for small z, and $O(|z|^{-1/2-p+\epsilon})$ for large z, where $p > 0$, uniformly in any angle interior to the above for any small positive ϵ, $0 < \epsilon < a$, and satisfying conditions (5.1) for all non-negative integers n on the interval $(-a/2 - Re\,(\alpha-\beta)/2\ ,\ a/2 - Re\,(\alpha-\beta)/2)$, if there exists such n.

6. Hankeltype and extended Hankel type transform in some other spaces of functions

Let Φ be any linear symmetric subspace of $L_1(R)$ or $L_2\,(R)$ having

$m(t) = 2^{it}\,\Gamma(2\alpha - it/2)/\,\Gamma(2\alpha - it/2)\ ,\ Re\,(\alpha-\beta) \neq -1, -3, \ldots ,$

as multiplier (symmetric means that if $\phi(t) \in \Phi$ then $\phi(-t) \in \Phi$). The multiplier $2^{it}\,\Gamma(2\alpha + it/2)/\Gamma(2\alpha - it/2)$ is infinitely differentiable and uniformly bounded on R,its derivatives grow slowly, therefore many classical spaces on R are special cases of Φ(for example, any L_1 or L_2 space with $L_\infty -$ weights, space $S(R)$ and space of infinitely differentiable functions with compact supports [30]). We define by $\mathcal{M}^{-1}(\Phi)$ the space of all functions g on R_+ that can be represented in the form

$$g(x) = \int_{-\infty}^{\infty} \phi(t) x^{it-1/2} dt, \qquad (6.1)$$

almost everywhere, where $\phi \in \Phi\big(if\ \phi \notin L_1(R)\big)$ the integral should be understood in L_2 –meaning. The space $\mathcal{M}^{-1}(L)$([28]) as well as the space of functions considered in Corollary 2 are special cases of $\mathcal{M}^{-1}\,(L)$([28]) as well as the space of functions considered in Corollary 2 are spaceial cases of $\mathcal{M}^{-1}\,(\Phi)$.

Theorem 6.1

The Hankel type transform (1.1) and the extended Hankel type transform (1.3) are bijectionsin $\mathcal{M}^{-1}(\Phi)$.

Proof

From (6.1) we see that $g \in \mathcal{M}^{-1}(\Phi)$ if and only if g can be expressed in the form of the inverse Mellin transform[27]

$$g(x) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} g^*(s) x^{-s}\, ds\ , \qquad (6.2)$$

Where $g^*(1/2 + it) \in \Phi$. Let $g^*(s)$ belong either to $L_1(1/2 - i\infty,\ \ 1/2 + i\infty)$ or $L_2(1/2 - i\infty, 1/2 + i\infty)$, and $g(x)$ be its inverse Mellin transform. It is proved (see [27]) for the case $g^*(s) \in L_2(1/2 - i\infty,\ 1/2 + i\infty)$ and [28] for the case $g^*(s) \in L_1(1/2 - i\infty,\ \ 1/2 + i\infty)$ that under these assumptions the Parseval formula for the Hankel type transform

$$(\mathcal{H}_{\alpha,\beta}g)(x) = \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\, g(y)\, dy \qquad (6.3)$$

$$= \frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} 2^{s-\frac{1}{2}}\frac{\Gamma(\alpha+s/2)}{\Gamma(\alpha-s/2)} g^*(1-s) x^{-s}\, ds, \qquad Re\,(\alpha-\beta) > -1,$$

holds. The Parseval formula (6.3) is also valid if we replace the Bessel type function $J_{\alpha-\beta}(x)$ by the truncated Bessel type function $J_{\alpha-\beta,m}(x)$ in case $1 - 2m\ > Re\,(\alpha-\beta) > \ -2m - 1$:

$$(\mathcal{H}_{\alpha,\beta}g)(x) = \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta,m}(xy)\, g(y)\, dy \qquad (6.4)$$

$$= \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} 2^{s-1/2}\frac{\Gamma(\alpha+s/2)}{\Gamma(2\alpha+\beta-s/2)} g^*(1-s) x^{-s}\, ds\,.$$

In fact, it is true if $g(x) \in L_2(R_+)$, or equivalently $g^*(s) \in L_2\left(\frac{1}{2} - i\infty,\ \frac{1}{2} + i\infty\right)$. Let now $g^*(s) \in L_1\left(\frac{1}{2} - i\infty,\ \frac{1}{2} + i\infty\right)$.From (1.4) we can infer that $J_{\alpha-\beta,m}(x) = O\left(x^{\alpha-\beta+2m}\right)$ at 0, therefore, integral $\int_0^1 x^{s-\frac{1}{2}} J_{\alpha-\beta,m}(x)\, dx$ converges absolutelyif $Res = \frac{1}{2}$.

Integral

$$\int_1^\infty x^{s-\frac{1}{2}} \sum_{k=0}^{m-1} \frac{(-1)^k (x/2)^{\alpha-\beta+2k}}{\Gamma(\alpha-\beta+k+1)k!}\, dx$$

Also converges absolutely. By applying asymptotic (2.3) of the Bessel type function one can easily see that integral $\int_1^N x^{s-\frac{1}{2}} J_{\alpha-\beta}(x)\, dx$is uniformly bounded forall $N \in [1,\infty)$and $s \in \left(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty\right)$. Hence integral

$$\int_0^N x^{s-\frac{1}{2}} J_{\alpha-\beta,m}(x)\, dx \qquad (6.5)$$

Is uniformly bounded for all $N \in [1,\infty)$and $s \in \left(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty\right)$.Consequently, by applying the result from [28] we can obtain (6.4).

Since $m^{-1}(-t) = m(t)$, then $2^{it}\frac{\Gamma(\alpha+it/2)}{\Gamma(\alpha-it/2)} g^*(1/2 - it)$ belongs to Φ if and only if $g^*(1/2 + it)$ belongs to Φ. Hence from (6.3) and (6.4) we obtain that $(\mathcal{H}_{\alpha,\beta} g)(x) \in \mathcal{M}^{-1}(\Phi)$ if and only if $g(x) \in \mathcal{M}^{-1}(\Phi)$. Thus Theorem 6.1 is proved.

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