Available online at www.elixirpublishers.com (Elixir International Journal)

Applied Mathematics

Elixir Appl. Math. 77 (2014) 28798-28801

ISSN: 2229-712X

b-chromatic number of some operations on Cycle and Path

D.Vijayalakshmi and K.Thilagavathi

Department of Mathematics, Kongunadu Arts and Science College, Coimbatore – 641 029.

ARTICLE INFO

Article history:

Received: 5 June 2014;

Received in revised form: 19 November 2014;

Accepted: 29 November 2014;

Keywords

b-chromatic number, b-coloring, Cycle, Path, Addition, Deletion, Union, Complement.

ABSTRACT

A b-vertex coloring of a graph[15] G is a proper vertex coloring of G such that each color class contains a vertex that has at least one vertex in every other color class in its neighborhood. The b-chromatic number of a graph G is the largest integerφ(G) for which G has a b-vertex coloring withφ(G) colors. This concept was introduced in [2] by Irving and Manlove by a certain partial ordering on all proper colorings in contrast to chromatic number χ (G),namelyχ (G) is the minimum of colors used among all minimal elements of this partial ordering, while φ(G) is the maximum of colors used among all minimal elements of the same partial ordering. The b-chromatic number has been considered with respect to subgraphs in [10,11], while the b-chromatic number under graph operations was considered in [15] for the Cartesian product and in [8] for the other three standard products. Operations on graphs produce new ones from older ones. Here the paper deals with the b-chromatic number of adding parallel chords in Cycle, Union of Path with Cycle and its complement, deletion and addition of vertices and edges in a Cycle.

1.1 Introduction

All graphs considered here are finite and simple. Notations and terminology not defined here will conform to those in [1]. For a graph *G*, let *V (G), E(G), p(G), q(G)* and $\bar{G}$, respectively, be the set of vertices, the set of edges, the order, the size and the complement of *G*.Let G be a simple graph and suppose that we have a proper coloring of G forwhich there exists a color class c such that every vertex v in c is not adjacentto any vertex in at least one other color class; then we can separately changethe color of each vertex in c to obtain a proper coloring with fewer Colors.Since then the b-chromatic number has drawn quite some attention among the scientific community. AlreadyIrving and Manlove [2] have shown, that computing φ(G) is an NP-complete problem in general.

The b-chromatic number has drawn much attention in scientific area [5,6,7,8,9,10]. We can easily imagine the colour classes as different communities, where every community *i* has a representative that is able to communicate with all the others communities. Even though the b-chromatic number is a simple concept, it is hard to determine the exact values, even for known families of graphs. This lead to studies of lower and upper bounds, [13].

The b-chromatic number has been considered with respect to subgraphs in [11], while the b-chromatic number under graph operations was considered in [15] for the Cartesian product and in [8] for the other three standard products.

Operations on graphs produce new ones from older ones. Unary operations create a new graph from the old one. It creates a new graph from the original one by a simple or a local change, such as addition or deletion of a vertex or an edge, merging and splitting of vertices, edge contraction,etc.

Definition 1.1 .1

A ***Chord***of a cycle C is an edge not in C whose end vertices lie in C.

Tele:

E-mail addresses: vijikasc@gmail.com

Definition 1.1.2

The Disjoint union of graphs [31, 45,] sometimes referred as simply graph union, which is defined as follows. Given two graphs G_1 and G_2, their union will be a graph such that $V(G_1 \cup G_2)= V(G_1) \cup V(G_2)$and $E(G_1 \cup G_2)= E(G_1) \cup E(G_2)$.

Definition 1.1.3

The **Complement** $\bar{G}$[31, 82] of a graph *G* is defined as a simple graph with the same vertex set as *G* and where two vertices *u* and *v* are adjacent only when they are not adjacent in *G*.

Definition 1.1.4

A ***closed walk*** with at least one edge in which no vertex except the terminal vertices appears more than once is called a ***cycle*** or ***circuit***.

Definition 1.1.5

A cycle that has odd length is an ***odd cycle***; otherwise it is an ***even cycle***. A graph is ***acyclic*** if it contains no cycles; unicyclic if it contains exactly one cycle

1.2 b-Chromatic Number of a Graph in Addition of Parallel Chords

1.2.1 Theorem

For any Cycle C_n, addition of parallel chords between non adjacent vertices holds the following statements:

- When *n* is odd, there exists a unique *3* cycle and $\left[\frac{n}{3}\right]$ times *4* cycle.
- When *n* is even, there exists exactly two *3* cycle and $\left[\frac{n}{2}\right]$- *2* times *4* cycle.

Proof

Let C_n be a Cycle with *n* vertices. Let $v_1,v_2,....v_n$ be the vertices and $e_1,e_2,...e_k$ be the edges of the Cycle C_n with parallel chords, where *k* is defined as

$$k = \begin{cases} \left[\frac{4n}{3}\right] & if\ n\ is\ odd \\ \left[\frac{4n}{3}\right] + 1 & if\ n\ is\ even \end{cases}$$

Case 1

When n is an odd cycle, the mutually adjacent edges e_1,e_n,e_{n+1} forms a 3 cycle and the remainingvertices are even, which forms a $\left\lceil\frac{n}{3}\right\rceil$times 4 cycles. Thus, when n is odd there exists a unique 3 cycle and $\left\lceil\frac{n}{3}\right\rceil$ times 4cycle.

Example

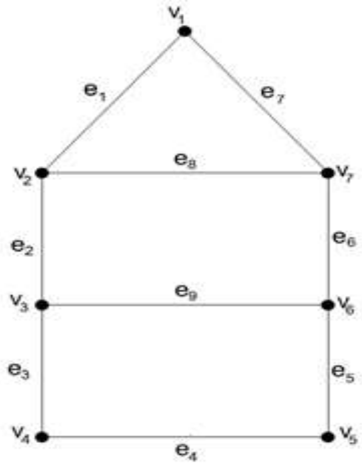

Figure 1: C_7 withparallel chords

Case 2

When $n$is an even cycle, the vertices with minimum degree forms a 3 cycle and vertices with maximum degree forms a 4 cycle. Thus for every even cycle there exist exactly two3 cycle and $\left\lceil\frac{n}{2}\right\rceil$ - 2 times 4 cycle.

Example

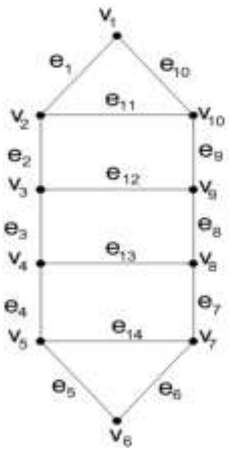

Figure 2: C_{10} with parallel chords

Theorem

The Cycle C_n with parallel chords has the b-Chromatic number four for every $n\geq 8$.

Example

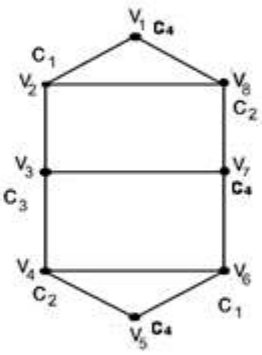

Figure 3: C_8 with parallel chords

1.3 b-Chromatic Number of a Graph when an Edge is removed

1.3.1 Theorem

For any Cycle ($n \geq 5$)with an edge $e \in V(C_n)$, $\varphi(C_n) = \varphi(C_n – e)$

Proof

Let C_n be the Cycle of length n. Let $v_1,v_2,...,v_n$ be the vertices arranged in anticlockwise direction i.e. $V(C_n)=\{v_1,v_2,....,v_n\}$ and the edge set be denoted as $E(C_n)=\{e_1,e_2,e_3,...e_n\}$. Here the vertex v_i is adjacent with the vertices v_{i-1} and v_{i+1} for $i=2,3,...n-1$, v_1 is adjacent with v_2,v_n and the vertex v_n is adjacent with v_{n-1} and v_1. We know that every Cycle is a connected graph with n vertices. It is evident that b-chromatic number of Cycle of length n for $n\geq 5$ is 3. Suppose if we delete any edge from the Cycle, we obtain a Path graph of length $n-1$with b-chromatic number 3.

Therefore $\varphi(C_n) = \varphi(C_n – e)$for every $n \geq 5$.

Example

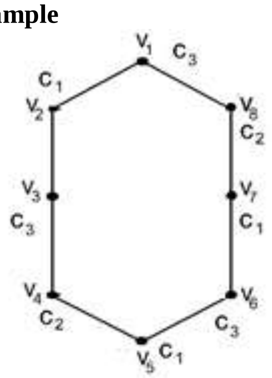

Figure 4(a):$\varphi(C_8)=3$

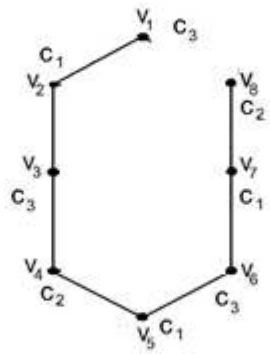

Figure 4(b): $\varphi(C_8\text{-}e)=3$

1.3.2 Corollary

$\varphi(C_n) \neq \varphi(C_n – e)$for every $n \leq 3$.

Example

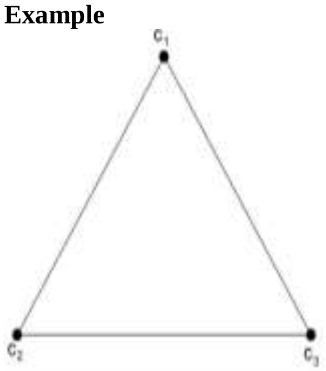

Figure 5(a): C_3

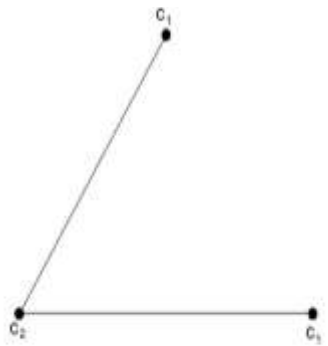

Figure 5(b):C_3-e

1.3.3 Corollary

For any Path for $n \geq 5$, $e \in V(P_n)$, $\varphi(P_n) \neq \varphi(P_n – e)$

1.4 b-Colouring of Adding a Pendant Vertex to each Vertex of a Cycle

1.4.1 Theorem

For any $n \geq 6$, $\varphi(C_n \bullet K_1) = \varphi(W_n)$

Proof

Let $v_i$for $1 \leq i \leq n$ are the vertices taken in the anticlockwise direction in the wheel graph W_n, where v_n is the hub. It is clear that the vertex v_i is adjacent withthe vertices v_{i-1} and v_{i+1} for $i=2,3,...n-1$, the vertex v_1 is adjacent with v_2 and v_{n-1}, the vertex $v_n$is adjacent with all the vertices. Here every vertex except the hub is incident with three edges, so we assign four colours, which produces a maximum and b-chromatic colouring by the colouring procedure. Also we know that the b-chromatic number of any Cycle has three coloursfor $n \geq 5$. If we attach a pendant vertex to every vertex of Cycle C_n, it is obvious that it has four coloursfor producing a b-chromatic colouring.

Therefore $\varphi(C_n \bullet K_1) = \varphi(W_n)$

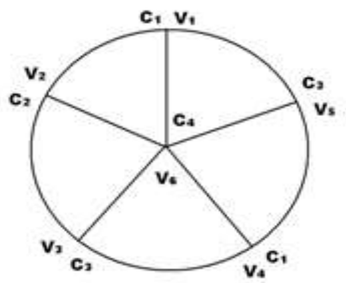

Figure 6(a): φ(C6 K1) = 4

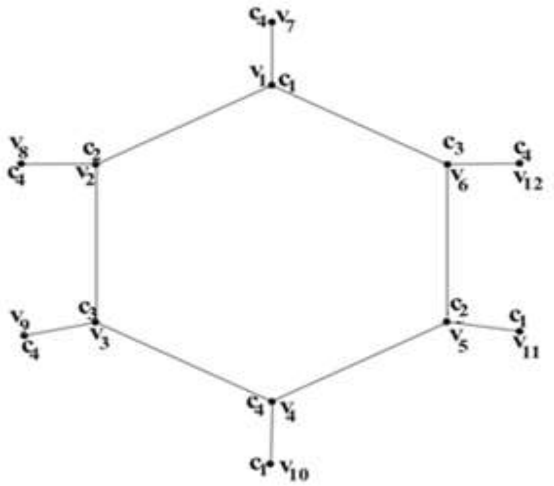

Figure 6(b): $\varphi(W_6) = 4$

1.4.2 Results obtained by Removing Edges from the Complete Graph

- $\varphi(K_4 - e) = \varphi(C_5) = \varphi(K_{1,n,n}), (n \geq 2)$
- $\varphi(K_3 - e) = \varphi(C_2) = \varphi(K_{1,n}), (n \geq 2)$
- $\varphi(K_5 - 2e) = \varphi(W_n), (n \geq 6)$
- $\varphi(K_5 - 3e) = \varphi(C_n), (n \geq 5)$

1.5 b-Chromatic Number of Union of Path with Cycle

1.5.1 Theorem

For any Path P_n and the Cycle C_n with n vertices, the b-chromatic number of $\overline{P_n U C_n}$ is given by $\varphi[\overline{P_n U C_n}] = n-1$ for $n \geq 2$.

Proof

Let $G_1 = P_n$ be a Path graph with n vertices and $n-1$ edges and $G_2 = C_n$ be a Cycle with n vertices and n edges. Let $G = G_1 \cup G_2$ be the graph obtained by the union of subgraph P_n and C_n of a graph has the vertex set $V(P_n) \cup V(C_n)$ and edge set $E(P_n) \cup E(C_n)$.

Consider $G = P_n \cup C_n$ whose vertex set $V(G) = \{v_1, v_2, v_3 \ldots v_{2n-2}\}$. Here in $P_n \cup C_n$, we see that the vertex v_i is adjacent with the vertices v_{i+1} and v_{i-1} for $i = 2,3 \ldots n-1, n+1, \ldots 2n-2$, v_1 is adjacent with v_2, v_{2n-2} and the vertex v_n is adjacent with the vertices v_1, v_{n-1} and v_{n+1}.

Now consider the graph $G = \overline{G_1 U G_2}$. By the definition of Complement, for any graph G, the non-adjacent vertices are adjacent in its complement. Here $\overline{G_1 U G_2}$ contains $2n-2$ vertices as in $G_1 \cup G_2$. Arrange the vertices of $\overline{G_1 U G_2}$ namely $v_1, v_2, v_3 .. v_n, v_{n+1}, \ldots v_{2n-2}$ in clockwise direction.

Assign a proper colouring to these vertices as follows. Consider the colour class $C = \{c_1, c_2, c_3 .. c_{n-1}\}$. First assign the colour c_i to the vertex v_i for $i = 1,2 \ldots 2n-2$, it will not produce a b-chromatic colouring, due to the above mentioned non-adjacency condition. Hence to make the colouring as b-chromatic one, assign the colour $C\left[\frac{i+1}{2}\right]$ to the vertices v_i and v_{i+1} for $i = 1,3,5 \ldots 2n-3$. Now all the vertices v_i for $i = 1,2 \ldots 2n-2$ realizes its own colour, which produces a b-chromatic colouring. Furthermore it is the maximum colouring possible.

Example

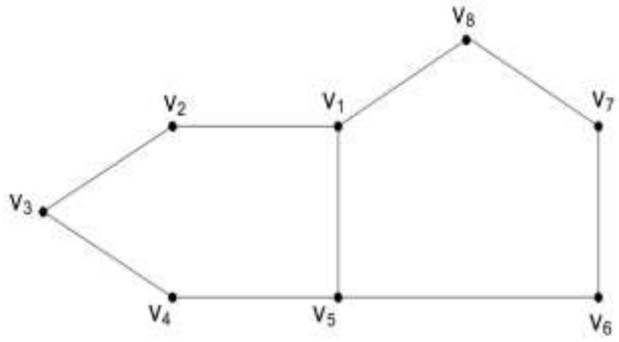

Figure 7(a): $P_5 \cup C_5$

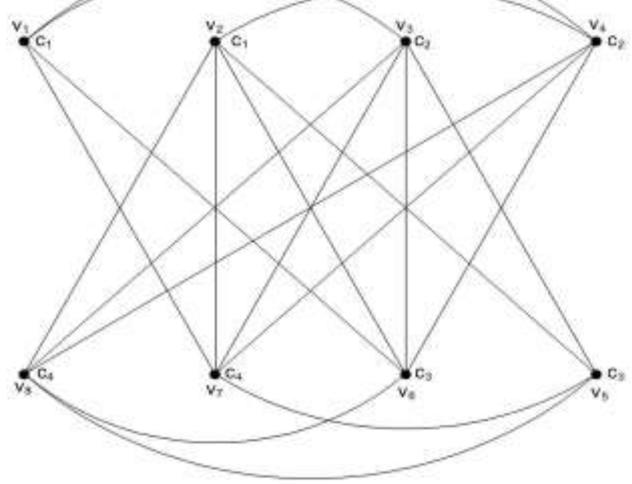

Figure 7(b): $\overline{P_5 U C_5}$ = 4

1.5.2 Theorem

$\varphi(P_n \cup C_n) = 3$ for every $n \geq 3$

Proof

The result is trivial from the above theorem.

1.5.3 Theorem

For any Path graph P_n and Cycle C_m with n and m vertices respectively, then $\varphi(P_n \cup C_m) = 3$ for $n \geq 2$.

Example

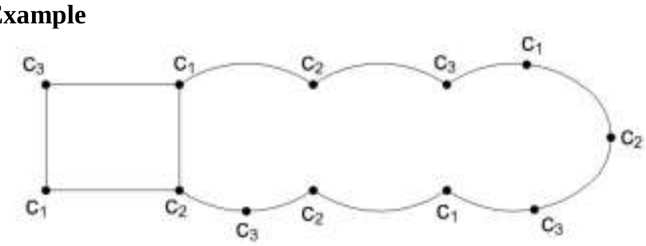

Figure 8: $P_4 \cup C_{10}$ =3

1.5.4 Result

For any integer $n > 2$, $\varphi(C_n) = \varphi(\overline{C_n})$

Example

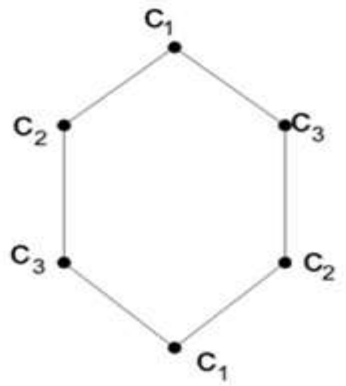

Figure 9(a): φ(C_6)=3

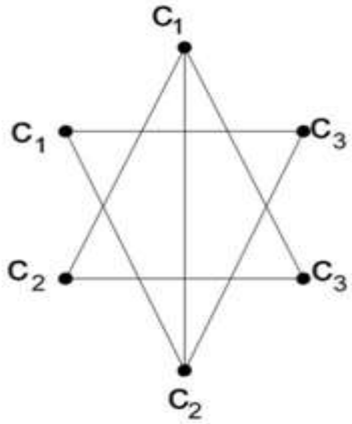

Figure 9(b): φ($\overline{C_6}$)=3

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, North-Holland, Amsterdam. [2]R.W. Irving, D.F. Manlove, The b-chromatic number of a graph, Discrete Appl. Math. 91 (1999) 127–141.

[3] M. Jakovac, S. Klavzar, The b-chromatic number of cubic graphs, Graphs Combin. Volume 26 (2010), pp: 107–118.

[4] M. Jakovac, I. Peterin, On the b-chromatic number of some products, Studia Sci. Math. Hungar. 49 (2012) 156–169.

[5] M. Jakovac, S. Klavzar, The b-chromatic number of cubic graphs, Graphs Combin. Volume 26 (2010), pp: 107–118.

[6] M.Kouider and M.Zaker, Bounds for the b-chromatic number of some families of graph, Discrete Mathematics, Volume 306, (2006), pp:617-623.

[7] M. Kouider, M.Maheo, Some bounds for the b-chromatic number of a graph, Discrete Mathematics, Volume 256, (2002), pp: 267–277.

[8] M. Kouider, M. Maheo, The b-chromatic number of the Cartesian product of two graphs, Studia Sci. Math. Hungar, Volume 44, (2007), pp: 49–55.

[9]M. Kouider, El Sahili, A, About b-colouring of regular graphs, Rapport de Recherche, No 1432, CNRS-University Paris Sud-LRI, 02/2006.

[10] M. Kouider, Mario Valencia-Pabon,On lower bounds for the b-chromatic number of connected bipartite graphs, Electronic Notes in Discrete Mathematics, Volume 37, (2011), pp: 399-404.

[11] M. Kouider, b-chromatic number of a graph, subgraphs and degrees, Rapportinterne LRI 1392.

[12] M. Kouider, A. El Sahili, About b-colouring of regular graphs, Rapport de Recherche No 1432, CNRS-Universite Paris Sud–LRI.

[13] M. Kouider, M. Maheo, Some bounds for the b-chromatic number of a graph, Discrete Math. 256 (2002) 267–277.

[14] M. Kouider, M. Maheo, The b-chromatic number of the Cartesian product of two graphs, Studia Sci. Math. Hungar. 44 (2007) 49–55.

[15] Marko Jakovac, IztokPeterin, On the b-chromatic number of Strong, lexicographic and strong product, (2009), University of Maribor.